Baron Münchhausen's Sequence

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Abstract

We investigate a coin-weighing puzzle that appeared in the all-Russian math Olympiad in 2000. We liked the puzzle because the methods of analysis differ from classical coin-weighing puzzles. We generalize the puzzle by varying the number of participating coins, and deduce a complete solution—perhaps surprisingly, the objective can be achieved in no more than two weighings regardless of the number of coins involved.

1 Introduction

The following coin-weighing puzzle, due to Alexander Shapovalov, appeared in the Regional round of the all-Russian math Olympiad in 2000 [2].

> Eight coins weighing $1, 2, \ldots, 8$ grams are given, but which weighs how much is unknown. Baron Münchhausen claims he knows which coin is which; and offers to prove himself right by conducting one weighing on a balance scale, so as to unequivocally demonstrate the weight of at least one of the coins. Is this possible, or is he exaggerating?

We chose to investigate this puzzle partly because classical coin-weighing puzzles [1] tend to ask the person designing the weighings to discover something they do not know, whereas here the party designing the weighings knows everything, and is trying to use the balance scale to convince someone else of something. There is therefore a difference of method: when the weigher is an investigator, they typically find themselves playing a minimax game against fate: they must construct experiments all of whose possible outcomes are as equally likely as possible, in order to learn the most they can even in the worst case. The Baron, however, knows everything, so he has the liberty to construct weighings whose results look very surprising from the audience's perspective.

We invite the reader to experience the enjoyment of solving this puzzle for themselves before proceeding; we will spoil it completely on page 4.

1.1 The Sequence

We will generalize this puzzle to n coins that weigh $1, 2, \ldots, n$ grams. We are interested in the minimum number of weighings on a balance scale that the Baron needs in order to convince his audience about the weight of at least one of those coins. It turns out that the answer is never more than two; over the course of the paper, we will prove this, and determine in closed from which n require two weighings, and which can be done in just one.

1.2 The Roadmap

In Section 2 we define the Baron's sequence again and show some of the flavor of this problem by calculating the first few terms. In Section 3 we prove the easy but perhaps surprising observation that this sequence is bounded; in fact that no term of this sequence can exceed three. That theorem opens the door to a complete description of all the terms of Baron Münchhausen's sequence, which we begin in Section 4 by explicitly finding the terms that are equal to one [8], to wit, the numbers n of coins such that the Baron can prove the weight of one coin among n in just one weighing.

Discriminating between two weighings sufficing and three being necessary is harder. The remainder of the paper is dedicated to proving that three is not a tight upper bound; namely that the Baron can always demonstrate the weight of at least one coin among any n in at most two weighings. Section 5 serves as a signpost by restating the theorem, and Section 6 briefly introduces some notation we will use subsequently. The actual proof is sufficiently involved that we break it down into a separate Section 7 for preliminaries, and then the proof itself in Section 8.

Finally, we close with Section 9 for some generalizations and ideas for future research.

2 Baron Münchhausen's Sequence

Baron Münchhausen's sequence $a(n)$ is defined as follows:

> Let n coins weighing $1, 2, \ldots, n$ grams be given. Suppose Baron Münchhausen knows which coin wieghs how much, but his audience does not. Then $a(n)$ is the minimum number of weighings the Baron must conduct on a balance scale, so as to unequivocally demonstrate the weight of at least one of the coins.

The original Olympiad puzzle is asking whether $a(8) = 1$.

2.1 Examples

Let us see what happens for small indices.

If $n = 1$ the Baron does not need to prove anything, as there is just one coin weighing 1 gram.

For $n = 2$ one weighing is enough. Place one coin on the left cup of the scale and one on the right, after which everybody knows that the lighter coin weighs 1 gram and the heavier coin weighs 2 grams.

For $n = 3$, the Baron can put the 1-gram and 2-gram coins on one cup and the 3-gram coin on the other cup. The cups will balance. The only way for the cups to balance is for the lone coin to weigh 3 grams.

For $n = 4$, one weighing is enough. The Baron puts the 1-gram and 2-gram coins on one cup and the 4-gram coin on the other cup. The only way for one coin out of four to be strictly heavier than two others from the set is for it to be the 4-gram coin. The 3-gram is also uniquely identified by the method of elimination.

For $n = 5$, Baron Münchhausen cannot do it in one weighing. This example is small enough to prove exhaustively: every possible outcome of every possible weighing admits of multiple assignments of weights to coins, which do not fix the weight of any one coin. For an example of the reasoning, suppose the Baron puts 1 coin on one cup and 2 coins on another, and shows that the single coin is heavier. All of $1+2<4$, $2+1<5$, and $1+3<5$ are consistent with this data, so no coin is uniquely identified. Checking all the other cases, as usual, is left to the reader.

For $n=6$, one weighing is enough. This case is similar to the case of $n=3$: $1+2+3=6$.

For $n=7$, one weighing is enough. This case is similar to the case of $n=4$: $1+2+3<7$.

For $n=8$, the original Olympiad problem asks whether one weighing is enough. It is—Baron Münchhausen can convince his audience by placing all the coins weighing 1 through 5 grams on one cup of the scale, and the two coins weighing 7 and 8 grams on the other. The only way for two coins with weights from the set $1, 2, \ldots, 8$ to balance five is for the two to weigh the most they can, at $7+8=15$ grams, and for the five to weigh the least they can at $1+2+3+4+5=15$ grams. This arrangement leaves excactly one coin off the scale, whose weight, by elimination, must be 6 grams.

So Baron Münchhausen's sequence begins with 0, 1, 1, 1, 2, 1, 1, 1. It also turns out that these examples illustrate all the methods of proving the weight of one coin with just one weighing; we will prove this in Section 4.

3 Three Weighings are Always Enough

In most of the coin problems we remember from childhood [1] the number of weighings needed to solve the problem grows logarithmically with the number of coins. Thus, our upper bound theorem may come as a surprise:

Theorem 3.1. *$a(n) \leq 3$.*

Before going into the proof we would like to introduce a little notation. First, we denote the x-th triangular number $x(x+1)/2$ by T_x.

We already did this in our example weighings, but we would like to make official the fact that, when describing weighings the Baron should carry out, we will denote a coin weighing i grams with just the number i. In addition, we will use round brackets to denote one coin of the weight indicated by the expression enclosed in the brackets. We need this notation to distinguish $i+1$, which represents two coins of weight i and 1 on some cup, from $(i+1)$, which represents one coin of weight $i+1$ on some cup.

Proof. We know, since Carl Friedrich Gauss proved it in 1796, [4, 6] that any number n can be represented as a sum of not more than 3 triangular numbers. Let $n = T_i + T_j + T_k$, where $T_i \leq T_j \leq T_k$ are triangular numbers with indices i, j, and k.

Barring special cases, Baron Münchhausen can display a sequence of three weighings with one coin on the right cup each: first

$$1 + 2 + 3 + \cdots + k = T_k,$$

then

$$T_k + 1 + 2 + 3 + \cdots + j = (T_k + T_j),$$

and finally

$$(T_k + T_j) + 1 + 2 + 3 + \cdots + i = n.$$

The first weighing demonstrates a lower bound of T_k on the weight of the coin the Baron put on the right. Since he then reuses that coin on the left, the second weighing demonstrates a lower bound of $T_k + T_j$ on the weight of the coin that goes on the right in the second weighing. Since he then reuses *that* coin in the third weighing, the audience finds a lower bound of $T_k + T_j + T_i$ on the coin on the right hand side of the last weighing. But since there are only n coins, there is already an upper bound of n on that coin's weight, so the assumed equality $T_k + T_j + T_i = n$ determines that coin's weight completely (as well as the weights of the T_k- and $(T_k + T_j)$-gram coins).

The Baron should start with the largest triangular number to make sure that he will not need any coin to appear in the same weighing twice: since k is the largest index, the coin T_k will not be in the sequence $1, 2, \ldots, j$, and the coin $(T_k + T_j)$ will not be in the sequence $1, 2, \ldots, i$.

When does this procedure fail? If $i = 0$, the last weighing is impossible but also redundant, because it would be asking to weigh the n-gram coin against itself. If $j = 0$, the second weighing is likewise impossible and redundant. If $k = 0$, there is nothing to prove because $n = 0$ as well. Finally, if $k = 1$, the first weighing is impossible because $T_k = 1$ occurs in $1 \ldots k$, but in this case n is at most 3, and that can be solved in fewer than three weighings anyway. □

4 When does One Weighing Suffice?

Let us look at where ones occur in this sequence. We characterize the possible weighings that could determine a coin by themselves. The determination of the numbers n of coins that admit such weighings, i.e., for which $a(n) = 1$, is a direct consequence.

4.1 One-Weighing Determinations

Theorem 4.1. *The weight of some coin can be confirmed with just one weighing if and only if all of:*

(i) one cup of that weighing contains all the coins with weights from 1 to some i;

(ii) the other cup contains all the coins with weights from some j to n;

(iii) Either the scale balances, or the cup containing the 1-gram coin is lighter by one gram; and

(iv) At least one cup contains exactly one coin, or exactly one coin is left off the scale.

Why can such a weighing be convincing? In general, i will be much larger than $n - j$. The only way for so few coins to weigh as much as (or more than) so many will be for the few to be the heaviest and the many to be the lightest. We show in the proof that those are exactly the convincing weighing structures; thereafter, in Section 4.2, we discuss the circumstances under which such a weighing exists and can therefore determine the weight of a single coin.

Proof. What does it mean for Baron Münchhausen to convince his audience of the weight k of some coin, using just one weighing? From the perspective of the audience, a weighing is a number of coins in one cup, a number of coins in the other cup, and a number of coins not on the scale, together with the result the scale shows (one or the other cup heavier, or both the same weight). For the audience to be convinced of the weight of some particular coin, it must therefore be the case that all possible arrangements of coin weights consistent with that data agree on the weight k of the coin in question.

"If" direction. Suppose all the conditions in the theorem statement are met. Then one cup of the scale contains i coins, and the other $n - j + 1$ coins. Suppose, for definiteness, that these are the left and right sides of the scale, respectively. The least that i coins can weigh is T_i. The most that $n - j + 1$ coins can weigh is $T_n - T_{j-1}$. The audience can compute these numbers; and if they are equal, then the audience can conclude that, for the scale to balance, the left cup must have exactly the coins of weights $1 \ldots i$, and the right cup must have exactly the coins of weights $j \ldots n$. Likewise, if $T_n - T_{j-1}$ exceeds T_i by one, the same allocation is the only way for the scale to indicate that the cup with i coins is lighter.

If either of the sets $1 \ldots i$ or $j \ldots n$ is a singleton, that determines the weight of that coin directly; otherwise, condition *(iv)* requires there to be only one coin off the scale, and the weight of that one remaining coin can be determined by the process of elimination.

"Only if" direction. Our proof strategy is to look for ways to alter a given arrangement of coin weights so as to change the weight given to the coin whose weight is being demonstrated; the requirement that all such alterations are impossible yields the desired constraints on convincing weighings.

First, obviously, the coin whose weight k the Baron is trying to confirm has to be alone in its group: either alone on some cup or the only coin not on the scale. After that observation we divide the proof of the theorem into several cases.

Case 1. The k-gram coin is on a cup and the scale is balanced. Then by above k is alone on its cup. We want to show two things: $k = n$, and the coins on the other cup weigh 1, 2, ..., i grams for some i. For the first part, observe that if $k < n$, then the coin with weight $k + 1$ must not be on the scale (otherwise it would overbalance coin k). Therefore, we can substitute coin $k + 1$ for coin k, and substitute a coin one gram heavier for the heaviest coin that was on the other cup, and produce thereby a different weight arrangement with the same observable characteristics but a different weight for the coin the Baron claims has weight k.

To prove the second part, suppose the contrary. Then it is possible to substitute a coin 1 gram lighter for one of the coins on the other cup. Now, if coin $k-1$ is not on the scale, we can also substitute $k-1$ for k, and again produce a different arrangement with the same observable characteristics but a different weight for the coin labeled k. On the other hand, if $k - 1$ is on the scale but $k - 2$ is not, then we can substitute $k - 2$ for $k - 1$ and then $k-1$ for k and the weighing is again unconvincing. Finally, if both $k-1$ and $k - 2$ are on the scale, and yet they balance k, then $k = 3$ and the theorem holds.

Consequently, $k = n = 1 + 2 + \cdots + i = T_i$ is a triangular number.

Case 2. The k-gram coin is on the lighter cup of the scale. Then: first, $k = 1$, because otherwise we could swap k and the 1-gram coin, making the light cup lighter and the heavy cup heavier or unaffected; second, the 2-gram coin is on the heavy cup and is the only coin on the heavy cup, because otherwise we could swap k with the 2-gram coin and not change the weights by enough to affect the imbalance; and finally $n = 2$ because otherwise we could change the weighing $1 < 2$ into $2 < 3$.

Thus the theorem holds, and the only example of this case is $k = 1, n = 2$.

Case 3. The k-gram coin is on the heavier cup of the scale. Then $k = n$ and the lighter cup consists of some collection of the lightest available coins, by the same argument as Case 1 (but even easier, because there is no need to maintain the balance). Furthermore, k must weigh exactly 1 gram more than the lighter cup, because otherwise, $k - 1$ is not on the lighter cup and can be substituted for k, making the weighing unconvincing.

Consequently, $k = n = (1 + 2 + \cdots + i) + 1 = T_i + 1$ is one more than a triangular number.

Case 4. The k-gram coin is not on a cup and the scale is not balanced. Then, since k must be off the scale by itself, all the other coins must be on one cup or the other. Furthermore, all coins heavier than k must be on the heavier cup, because otherwise we could make the lighter cup even lighter by substituting k for one of those coins. Likewise, all coins lighter than k must be on the lighter cup, because otherwise we could make the heavier cup even heavier by substituting k for one of those coins. So the theorem holds; and furthermore, the cups must again differ in weight by exactly 1 gram, because otherwise we could swap k with either $k - 1$ or $k + 1$ without changing the weights enough to affect the result on the scale.

Consequently, the weight of the lighter cup is $k(k - 1)/2$, and the weight of the heavier cup is $k(k - 1)/2 + 1$. Thus the total weight of all the coins is $n(n + 1)/2 = k(k - 1)/2 + k + (k(k - 1)/2 + 1) = k^2 + 1$. In other words, case 4 is possible iff n is the index of a triangular number that is one greater than a square.

Case 5. The k-gram coin is not on a cup and the scale is balanced. This case is hairier than all the others combined, so we will take it slowly (noting first that all the coins besides k must be on some cup).

Lemma 4.2. *The two coins $k - 1$ and $k - 2$ must be on the same cup, if they exist (that is, if $k > 2$). Likewise $k - 2$ and $k - 4$; $k + 1$ and $k + 2$; and $k + 2$ and $k + 4$.*

Proof. Suppose the two coins $k - 1$ and $k - 2$ are not on the same cup. Then we can rotate k, $k - 1$, and $k - 2$, that is, put k on the cup with $k - 1$, put $k-1$ on the cup with $k-2$, and take $k-2$ off the scale. This makes both cups heavier by one gram, producing a weighing with the same outward characteristics as the one we started with, but a different coin off the scale. The same argument applies to the other three pairs of coins we are interested in, mutatis mutandis. □

Lemma 4.3. *The four coins $k-1$, $k-2$, $k-3$ and $k-4$ must be on the same cup if they exist (that is, if $k \geq 5$).*

Proof. By Lemma 4.2, the three coins $k-1$, $k-2$, and $k-4$ must be on the same cup. Suppose coin $k-3$ is on the other cup. Then we can swap $k-1$ with $k-3$ and k with $k-4$. Each cup becomes lighter by 2 grams without changing the number of coins present, the balance is maintained, and the Baron's audience is not convinced. □

Lemma 4.4. *If coin $k-4$ exists, that is if $k \geq 5$, all coins lighter than k must be on the same cup.*

Proof. By Lemma 4.3, the four coins $k-1$, $k-2$, $k-3$ and $k-4$ must be on the same cup. Suppose some lighter coin is on the other cup. Call the heaviest such coin c. Then, by choice of c, the coin with weight $c+1$ is on the same cup as the cluster $k-1 \ldots k-4$, and is distinct from coin $k-2$. We can therefore swap c with $c+1$ and swap k with $k-2$. This increases the weight on both cups by 1 gram without changing how many coins are on each, but moves k onto the scale. The Baron's audience is again unconvinced. □

Lemma 4.5. *Theorem 4.1 is true for $k \geq 5$.*

Proof. By Lemma 4.4, all coins lighter than k must be on the same cup. Further, if a coin with weight $k+4$ exists, then by the symmetric version of Lemma 4.4, all coins heavier than k must also be on the same cup. They must be on the other cup from the coins lighter than k because otherwise the scale would not balance, and the theorem is true.

If no coin with weight $k+4$ exists, that is, if $n \leq k+3$, how can the theorem be false? All the coins lighter than k must be on one cup, and their total weight is $k(k-1)/2$. Further, in order to falsify the theorem, at least one of the coins heavier than k must also be on that same cup. So the minimum weight of that cup is now $k(k-1)/2 + k + 1$. But we only have at most two coins for the other cup, whose total weight is at most $k+2+k+3 = 2k+5$. For the scale to even have a chance of balancing, we must have

$$k(k-1)/2 + k + 1 \leq 2k + 5 \Leftrightarrow k^2 - 3k - 8 \leq 0.$$

Finding the largest root of that quadratic we see that $k < 5$.

So for $k \geq 5$, the collection of all coins lighter than k is heavy enough that either one needs all the coins heavier than k to balance them, or there are enough coins heavier than k that the theorem is true by symmetric application of Lemma 4.4.

Completion of Case 5. It remains to check the case for $k < 5$. If $n > k+3$, then coin $k+4$ exists. If so, all the coins heaver than k must be on the same cup. Furthermore, since k is so small, they will together weigh more than half the available weight, so the scale will be unbalanceable. So $k < 5$ and $n \leq k + 3 \leq 7$.

For lack of any better creativity, we will tackle the remaining portion of the problem by complete enumeration of the possible cases, except for the one observation that, to balance the scale with just the coin k off it, the total weight of the remaining coins, $n(n+1)/2-k$, must be even. This observation cuts our remaining work in half. Now to it.

Case 5; Seven Coins: $n = 7$**.** Then $5 > k \geq n - 3 = 4$, so $k = 4$. Then the weight on each cup must be 12. One of the cups must contain the 7 coin, and no cup can contain the 4 coin, so the only two weighings the Baron could try are $7+5 = 1+2+3+6$, and $7+3+2 = 1+5+6$. But the first of those is unconvincing because $k + 1 = 5$ is not on the same cup as $k + 2 = 6$, and the second because it has the same shape as $7 + 3 + 1 = 2 + 4 + 5$ (leaving out the 6-gram coin instead of the asserted 4-gram coin).

Case 5; Six Coins: $n = 6$**.** Then $5 > k \geq n-3 = 3$, and $n(n+1)/2 = 21$ is odd, so k must also be odd. Therefore $k = 3$, and the weight on each cup must be 9. The 6-gram coin has to be on a cup and the 3-gram coin is by presumption out, so the Baron's only chance is the weighing $6+2+1 = 4+5$, but that does not convince his skeptical audience because it looks too much like the weighing $1 + 3 + 4 = 6 + 2$.

Case 5; Five Coins: $n = 5$**.** Then $5 > k \geq n - 3 = 2$, and $n(n+1)/2 = 15$ is odd, so k must also be odd. Therefore $k = 3$, and the weight on each cup must be 6. The only way to do that is the weighing $5 + 1 = 2 + 4$, which does not convince the Baron's audience because it looks too much like $1 + 4 = 2 + 3$.

Case 5; Four Coins: $n = 4$**.** Then the only way to balance a scale using all but one coin is to put two coins on one cup and one on the other. The only two such weighings that balance are $1 + 2 = 3$ and $1 + 3 = 4$, but they leave different coins off the scale.

The remaining cases, $n < 4$, are even easier. That concludes the proof of Case 5.

Consequently, by an argument similar to the one in case 4 we can show that any number n of coins to which case 5 applies must be the index of a square triangular number. □

This concludes the proof of Theorem 4.1. □

4.2 The Indices of Ones

While proving the theorem we accumulated descriptions of all possible numbers of coins that allow the Baron to confirm a coin in one weighing. We collect that list here to finish the description of the indices of ones in Baron Münchhausen's sequence $a(n)$. The following list corresponds to the five cases in the proof of Theorem 4.1:

(i) n is a triangular number: $n = T_i$. Then the weighing $1+2+3+\ldots+i = n$ proves weight of the n-gram coin. For example, for six coins the weighing is $1 + 2 + 3 = 6$.

(ii) $n = 2$. The weighing $1 < 2$ proves the weight of both coins.

(iii) n is a triangular number plus one: $n = T_i + 1$. Then the weighing $1 + 2 + 3 + \ldots + i < n$ proves the weight of the n-gram coin. For example, for seven coins the weighing is $1 + 2 + 3 < 7$.

(iv) n is the index of a triangular number that is a square plus one: $T_n = k^2+1$. Then the weighing $1+2+3+\ldots+(k-1) < (k+1)+\ldots+n$ proves the weight of the k-gram coin. For example, the fourth triangular number, which is equal to ten, is one greater than a square. Hence the weighing $1 + 2 < 4$ can identify the coin that is not on the cup. The next number like this is 25, and the corresponding weighing is $1 + 2 + \cdots + 17 < 19 + 20 + \cdots + 25$.

(v) n is the index of a square triangular number: $T_n = k^2$. Then the weighing $1+2+3+\ldots+(k-1) = (k+1)+\ldots+n$ proves the weight of the k-gram coin. For example, we know that the 8th triangular number is 36, which is a square: our original problem corresponds to this case.

The sequence of indices of ones in the sequence $a(n)$ starts as: 1, 2, 3, 4, 6, 7, 8, 10, 11, 15, 16, 21, 22, 25, 28, 29, 36, 37, 45, 46, 49, 55, 56, 66, 67, 78, 79, 91, 92.

4.3 Discussion

If we have four coins, then the same weighing $1 + 2 < 4$ identifies two coins: the coin that weighs three grams and is not on the scale and the coin weighing four grams that is in a cup. The other case like this is for $n = 2$. Comparing the two coins to each other we can identify both of them. It is clear that there are no other cases like this. Indeed, for the same weighing to identify two different coins, it must be the n-gram coin on a cup, and the $(n-1)$-gram coin off the scale. From here we can see that n cannot be very big.

As usual, we want to give our readers something to think about. We have given you the list of four sequences that correspond to four cases describing all the numbers for which the Baron can prove the weight of one coin in one weighing. Does there exist a number greater than four that belongs to two of these sequences? In other words, does there exist a total number of coins such that the Baron can have two different one-weighing proofs for two different coins?

5 Two Weighings are Always Enough

Our main theorem states that Baron Münchhausen never needs three weighings, for two are always enough.

Theorem 5.1. $a(n) \leq 2$.

6 Notation

In the proofs of the previous theorems and lemmas, we have already seen some recurring elements: triangular numbers are important; contiguous ranges of coins are important. Additional common elements will arise in our further explorations, so we introduce some notation now to more easily manipulate them.

As before the coins are numbered according to their weights, and we will continue to use the number i to denote an i-gram coin on a cup, using round brackets as before to distinguish a single coin of a weight we need to compute from two separate coins. For example, (1+2) means the 3-gram coin, whereas 1+2 means the 1-gram and the 2-gram coin.

We will also continue to use the notation T_x to denote the xth triangular number $x(x+1)/2$.

We introduce the notation $[x \ldots y]$ for the set of all consecutive coins between x and y, inclusive; and we will occasionally construct weighings with set notation. Inside expressions in square brackets we will not parenthesize computations: $[3+4\ldots 11-1]$ is the set of coins weighing from 7 to 10, inclusive, and does not include the coins 3, 4, 11, or 1.

If A denotes a set of coins, then $|A|$ denotes the total weight of those coins (not the cardinality of the set).

When representing a weighing as an equality/inequality we will refer to the left and right sides of the equality/inequality as the left and right cups of the weighing, respectively.

7 Preliminaries

Before we proceed with the main section of the proof, we will prove two lemmas that we are going to need, and that will demonstrate the machinery we will use to prove the main Theorem 5.1.

Lemma 7.1. *If n, $n-1$, or $n-2$ is a sum of two triangular numbers, then the Baron can demonstrate the weight of the n-gram coin in two weighings.*

Proof. This is a direct corollary of the argument used to prove Theorem 3.1. If $n = T_a + T_b$, that argument applies exactly. In the other two cases, the Baron can make judicious use of unbalanced weighings.

If $n = T_a + T_b + 1$, for $a \leq b$, then one of the weighings needs to be unbalanced, for example

$$\begin{aligned} [1\ldots b] &= T_b \\ [1\ldots a] + T_b &< n. \end{aligned}$$

If $n = T_a + T_b + 2$, then both weighings should be unbalanced:

$$\begin{aligned} [1 \ldots b] &< (T_b + 1) \\ [1 \ldots a] + (T_b + 1) &< n. \end{aligned}$$

□

Since triangular numbers are pretty dense among the small integers, Theorem 4.1 and Lemma 7.1 account for many small n. This is good, because the main proof in Section 8 does not go through for small n. In particular, the reader is invited to verify that the smallest n that does not fall under the purview of Lemma 7.1 is $n = 54$; for example by consulting sequence A020756 in OEIS [3].

The following covers a special case we will encounter in the main proof, and coincidentally demonstrates the argument we will use in the main proof that the complicated weighings we will present will, in fact, convince the Baron's audience.

Lemma 7.2. *If there exists an a such that $2n = T_a + T_{a+1}$, the Baron can prove the weight of the n-gram coin in two weighings.*

Proof. We know that $T_a < n < T_{a+1}$ and, in fact, $n = T_a + \frac{a+1}{2}$. Suppose we can find coins x and y with $a + 1 < x < y = x + \frac{a+1}{2} < n$. Then the Baron can present the following two weighings:

$$[1 \ldots a] + y = x + n$$

and

$$[1 \ldots a + 1] + x = y + n.$$

They will balance by the choice of x and y. Why will they convince the Baron's audience?

Let the audience consider the sum of the two weighings. The coins x and y appear on both sides of the sum, so they do not affect the balance of the total. Besides them, a coins appeared twice on the left, and one additional coin appeared once on the left; and this huge pile of stuff was balanced by just two appearances of a single coin on the right. How is this possible? The least possible total weight of the left-hand sides (except x and y) occurs if the coins that appeared twice have weights $[1 \ldots a]$, and the coin that appeared once has weight $a + 1$, for a total weight of $T_a + T_{a+1}$. The greatest possible total weight of the right-hand sides (again excluding x and y) occurs if the solitary coin on the right weighs n grams. But the known fact that $2n = T_a + T_{a+1}$ guarantees that, even in this extreme case the scale will just barely balance; so any other set of weights would cause the left cup to overbalance the right in at least one of the weighings Baron Münchhausen conducts. Therefore, since, in fact, neither left cup overbalanced its corresponding right cup, the Baron's audience is forced to conclude that the solitary coin on the right must weigh n grams, as was the Baron's intention. (Coincidentally, this scenario also proves the weight of the $a + 1$ coin.)

Now, when can we find such coins x and y? We can safely take the $a + 2$ coin for x. Then the desired y coin will exist if $n > a + 2 + \frac{a+1}{2}$, which is equivalent to $T_a > a+2$, which holds for $a \geq 3$. The last condition translates into $n \geq 8$.

Smaller n are covered by Lemma 7.1. □

8 Proof of the Main Theorem

There are two magical steps. First, let $a \leq b \leq c$ be such that

$$T_a + T_b + T_c = n + T_n. \tag{1}$$

By the triangular number theorem, proved by Gauss in his diary, [4, 6] such a decomposition of $T_n + n$ into three triangular numbers is always possible. We should remark at this point that $c > n$ would imply $T_c \geq T_{n+1} > T_n + n$ so is impossible; and that $c = n$ would imply $T_c = T_n$ so $T_a + T_b = n$, allowing the Baron to proceed by the method in Lemma 7.1. So we can assume $c < n$.

Second, let us try to represent $T_c - n$ as the sum of some subset S of weights from the range $[a + 1 \ldots n - 1]$. Now there are three non-magical steps. We will prove that if such a representation exists, then the Baron can convince his audience of the weight of the n-gram coin in two weighings, by a particular method to be described forthwith; then we will take some time to study the properties of sums of subsets of ranges of integers; and then at the last we will systematically examine possible choices of a, b, and c, and prove that the above-mentioned subset S really does exist, except in one case, for which Lemma 7.2 supplies an alternate method of solution.

8.1 Step 1: What to do with S

Lemma 8.1. *Let a, b, and c satisfy*

$$T_a + T_b + T_c = n + T_n.$$

Let S be a subset of $[a+1 \ldots n-1]$ for which

$$|S| = T_c - n.$$

Then there exist two weighings that uniquely identify the n-gram coin.

Proof. Let $\bar{S}$ denote the complement of S in $[a+1 \ldots n-1]$. We want to make a useful weighing out of the assumption about S, so let us proceed as follows:

$$T_c = |S| + n,$$

which we rewrite in terms of weights of coins

$$[1 \ldots a] + [a+1 \ldots c] = S \cap [a+1 \ldots c] + S \cap [c+1 \ldots n-1] + n,$$

and cancel coins appearing on both sides to get

$$[1 \ldots a] + \bar{S} \cap [a+1 \ldots c] = S \cap [c+1 \ldots n-1] + n. \tag{2}$$

Observe that (2) now forms a legal weighing; and indeed, let us take it to be the first weighing.

Now we want to make another weighing that will, together with (2), demonstrate the weight of the n-gram coin. Let us begin by massaging (1):

$$\begin{aligned} T_a + T_b + T_c &= n + T_n \\ T_a + T_b &= T_{n-1} - T_c + 2n \\ T_a + T_b + |\bar{S} \cap [b+1 \ldots c]| &= |\bar{S} \cap [b+1 \ldots c]| + |[c+1 \ldots n-1]| + 2n. \end{aligned}$$

Now, converting the last equality into coins and subtracting (2), we get

$$[1 \ldots a] + S \cap [a+1 \ldots b] = \bar{S} \cap [b+1 \ldots c] + \bar{S} \cap [c+1 \ldots n-1] + n. \tag{3}$$

Again, each coin occurs at most once, so the Baron can legitimately take (3) as his second weighing.

We have just shown that (2) and (3) represent four sets of coins that can be weighed against each other in the indicated pattern, and that the scale will balance if they are. Now, why do these two weighings uniquely identify the n-gram coin? Consider which coins appear on which sides of those two equations. Let L_1 and R_1 be the left- and right-hand sides of the first weighing (2), respectively, and likewise L_2 and R_2 for the second weighing (3). Also, let O_1 and O_2 be the sets of coins that do not participate in (2) and (3), respectively. Then

$$\begin{aligned}
[1 \ldots a] &= L_1 \cap L_2, \\
\bar{S} \cap [a+1 \ldots b] &= L_1 \cap O_2, \\
S \cap [a+1 \ldots b] &= O_1 \cap L_2, \\
S \cap [b+1 \ldots c] &= O_1 \cap O_2, \\
\bar{S} \cap [b+1 \ldots c] &= L_1 \cap R_2, \\
\bar{S} \cap [c+1 \ldots n-1] &= O_1 \cap R_2, \\
S \cap [c+1 \ldots n-1] &= R_1 \cap O_2, \\
n &= R_1 \cap R_2.
\end{aligned}$$

Seeing the two weighings (2) and (3), Baron Münchhausen's audience reasons analagously to how they did in the proof of Lemma 7.2. They consider the sum of the two weighings, which tells them

$$|L_1| + |L_2| = |R_1| + |R_2|.$$

They see that some coins, namely $L_1 \cap R_2$, (which the Baron knows to be $\bar{S} \cap [b+1 \ldots c]$) appeared first on the left and then on the right, so those coins do not affect the balance of the sum. The audience also sees that

(i) a coins appeared on the left both times ($L_1 \cap L_2$);

(ii) $b-a$ coins appeared on the left once and never on the right ($(L_1 \cap O_2) \cup (O_1 \cap L_2)$);

(iii) $n-1-c$ coins appeared on the right once and never on the left ($(R_1 \cap O_2) \cup (O_1 \cap R_2)$); and

(iv) just one coin appeared on the right both times ($R_1 \cap R_2$).

Now, a and $b-a$ are going to be much bigger than $n-c-1$ and 1, so the audience will be surprised that so many coins can be balanced by so few. And they will wonder how to minimize the total weight

$$2|L_1 \cap L_2| + |(L_1 \cap O_2) \cup (O_1 \cap L_2)|$$

of the many, and how to maximize the total weight

$$|(R_1 \cap O_2) \cup (O_1 \cap R_2)| + 2|R_1 \cap R_2|$$

of the few. And they will see that to do this, they must

(i) let the coins in $L_1 \cap L_2$ have the weights $[1 \ldots a]$, as they occur on the left twice;

(ii) let the coins in $(L_1 \cap O_2) \cup (O_1 \cap L_2)$ have the weights $[a+1 \ldots b]$, as they occur on the left once;

(iii) let the coins in $(R_1 \cap O_2) \cup (O_1 \cap R_2)$ have the weights $[c+1 \ldots n-1]$, as they occur on the right once; and

(iv) let the sole coin in $R_1 \cap R_2$ have weight n, as it occurs on the right twice.

And then they will see from (1), which can be rewritten as

$$\begin{aligned} T_a + T_b + T_c &= T_n + n \\ T_a + T_b &= T_{n-1} - T_c + 2n \\ 2|[1 \ldots a]| + |[a+1 \ldots b]| &= |[c+1 \ldots n-1]| + 2n \end{aligned}$$

that even if they minimize the left and maximize the right, the scale will just barely balance. And then they will know that any other weights than those would have made the left heavier than the right, and since the scale did balance, those are the weights that must have been, and they will wonder in awe at the Baron's skill in convincing them of the weight of his chosen coin out of n in only two weighings. □

We have established that the existence of a subset S of $[a+1 \ldots n-1]$ that adds up to $|S| = T_c - n$ suffices to let the Baron convince his audience of the weight of the coin labeled n in two weighings. Now, when does such a subset reliably exist?

8.2 Step 2: Sums of subsets of ranges

To answer this question, let us study the behavior of sums of subsets of ranges of positive integers in general. The results of this segment probably generalize to negative integers and beyond, and are probably published in many books, but we decided that it is faster to derive them ourselves than drive to the library.

Suppose we have some range of integers $[s \dots t]$. What are the possible sums of its subsets? First, what are the possible sums of subsets of a fixed size, say k? Well, the smallest sum of k elements of $[s \dots t]$ is of course the sum of the k smallest elements of $[s \dots t]$:

$$s + (s+1) + \dots + (s + (k-1)) = ks + T_{k-1} = T_{s+k-1} - T_{s-1}.$$

The largest sum of k elements of $[s \dots t]$ is of course the sum of the k largest elements of $[s \dots t]$:

$$t + (t-1) + \dots + (t - (k-1)) = kt - T_{k-1} = T_t - T_{t-k}.$$

What is more, given any subset K of k elements of $[s \dots t]$ that are not the k largest, we can change one of them for an element one larger that was not in K, thus producing a subset whose sum is larger by one. Since we can walk all the way from the k smallest elements to the k largest elements by increments of one, the possible sums cover the whole range between the least and the greatest possible values, and we have just proven

Lemma 8.2. *The set of possible sums of subsets of size k of a range $[s \dots t]$ is exactly the range $[ks + T_{k-1} \dots kt - T_{k-1}]$.*

Now, what about the overall behavior of subsets of any size? Well, subsets of size k form a contiguous range, and subsets of size $k+1$ also form a contiguous range. Do those ranges join to form a larger range, or is there a gap? In other words, is one plus the maximum sum of subsets of size k a possible sum of subsets of size $k+1$? This will be true if and only if replacing the k largest elements of $[s \dots t]$ with the $k+1$ smallest does not increase the sum by more than 1, or

$$kt - T_{k-1} + 1 \geq (k+1)s + T_k. \tag{4}$$

Moreover, if turning the k largest elements into the $k+1$ smallest elements does not cause an increase exceeding 1, the same will hold for $k+1$, as long as k is less than the middle point of the segment $[s \dots t]$: $k < \frac{t-s}{2}$. Indeed if we sum up the inequality (4) with $t - k \geq s + k + 1$, we get

$$(k+1)t - T_k + 1 \geq (k+2)s + T_{k+1},$$

the condition that the next two ranges overlap.

This means that if the k-subset range overlaps the $k+1$-subset range, then all larger ranges will also overlap, at least until size $\frac{t-s}{2}$. Also, subsets of size greater than $\frac{t-s}{2}$ overlap symmetrically to their smaller counterparts, because the sum of any such subset is just the total sum of all numbers between s and t minus the sum of the complement of that subset. This demonstrates

Lemma 8.3. *If $k < (t-s)/2$ is such that*

$$kt - T_{k-1} + 1 \geq (k+1)s + T_k,$$

then the possible sums of subsets of sizes $[k \dots t-s-k]$ create a contiguous range. In other words, it is possible to find subsets of the range $[s \dots t]$ that sum up to any number between $ks + T_{k-1}$ and $(t-s-k)t - T_{t-s-k-1}$.

Considering the possibility that ranges may start overlapping from $k=1$, that is $t - T_0 + 1 \geq 2s + T_1$, leads us to

Corollary 8.4. *If $t+1 \geq 2s+1$, or, equivalently, $s \leq t/2$, the subsets of the range $[s \dots t]$ can achieve any sum in*

$$[s \dots T_t - T_s].$$

Considering the possibility that ranges may start overlapping from $k=2$, that is $2t - T_1 + 1 \geq 3s + T_2$, leads us to

Corollary 8.5. *If $2t \geq 3s+3$, or, equivalently, $s \leq \frac{2}{3}t - 1$, the subsets of the range $[s \dots t]$ can achieve any sum in*

$$[2s+1 \dots T_t - T_{s+1}].$$

These two facts will prove invaluable to characterizing when $T_c - n$, from above, can be achieved as the sum of some set of coins from a given range.

8.3 Step 3: Systematic study of possibilities for a, b, and c

We are now ready to finish this proof. Recall the setup: The Baron has n coins; we have made a decomposition into three triangular numbers

$$n + T_n = T_a + T_b + T_c, \quad a \leq b \leq c;$$

and we know that if we can find a subset S of $[a+1 \dots n-1]$ for which

$$T_c - n = |S|,$$

the Baron can convince his audience of the weight of the n-gram coin in two weighings. We also know, from the corollaries above, that

(i) If $2a + 3 \leq n$, sums of subsets of $[a+1 \dots n-1]$ include the range $[a+1 \dots T_{n-1} - T_{a+1}]$, and

(ii) If $3a+6 \leq 2n-2$, sums of subsets of $[a+1 \dots n-1]$ include the range $[2a+3 \dots T_{n-1} - T_{a+2}]$.

Here is the main idea for the remainder of this section, and with it, the proof: As T_a is the smallest number in our decomposition, we know that $T_a \leq \frac{T_n+n}{3} < \frac{T_{n+1}}{3}$. We can conclude from this that $a < \frac{n+1}{\sqrt{3}}$. Since $\frac{n+1}{\sqrt{3}}$ grows slower than $\frac{2n}{3}$, for large n we expect the condition $3a+6 \leq 2n-2$ in Corollary 8.5 to hold. Then it will suffice to prove that $T_c - n$ falls into the range $[2a+3 \dots T_{n-1} - T_{a+2}]$. In general, the lower bound will be easy; and for the upper bound we will find that if T_c is large, then T_a will be small, so this analysis will cover all but a few possibilities for c; but these few will be extreme enough and few enough to handle directly.

Now to it. Since $a < \frac{n+1}{\sqrt{3}}$,

$$3a + 6 < (n+1)\sqrt{3} + 6.$$

For $n \geq 37$,

$$(n+1)\sqrt{3} + 6 \leq 2n - 2.$$

Combining these two we get the desired

$$3a + 6 < 2n - 2,$$

so Corollary 8.5 applies. Subsets of $[a+1 \dots n-1]$ can take on all sums in the range $[2a+3 \dots T_{n-1} - T_{a+2}]$.

Does $T_c - n$ fall into this range? For the upper bound, we have the sequence of equivalent inequalities, starting with the desired one

$$\begin{aligned} T_c - n &\leq T_{n-1} - T_{a+2} \\ T_a + T_c + (a+1) + (a+2) &\leq T_n \\ T_a + T_c + (a+1) + (a+2) + n &\leq T_n + n = T_a + T_b + T_c \\ n + 2a + 3 &\leq T_b. \end{aligned}$$

On the other hand, if $n + 2a + 3 \le T_b$, then

$$T_c - n \ge T_b - n \ge 2a + 3. \tag{5}$$

So for $n \ge 37$, as long as $T_b \ge n+2a+3$, a subset S of $[a+1 \dots n-1]$ can be found that sums to $T_c - n$, permitting the Baron to convince his audience of the weight of the coin labeled n.

When can we guarantee that $T_b \ge n + 2a + 3$? We know that $a < \frac{n+1}{\sqrt{3}}$, so it is enough to guarantee that

$$T_b \ge n + \frac{2(n+1)}{\sqrt{3}} + 3.$$

As $T_b \ge T_a$, it is enough to guarantee that

$$T_a + T_b = T_n - T_c + n \ge 2n + \frac{4(n+1)}{\sqrt{3}} + 6.$$

If $c \le n-4$, then $T_n - T_c + n \ge 5n-6$. For $n \ge 37$, $5n-6 > 2n + \frac{4(n+1)}{\sqrt{3}} + 6$, so $T_c - n$ does, in fact, fit into the desired range.

It now remains to analyse the cases when $c > n-4$. As remarked earlier, $c > n$ is impossible, and $c = n$ implies that $n = T_a + T_b$, so two weighings suffice by Lemma 7.1. So we are left with three cases: $c = n-1$, $c = n-2$ and $c = n-3$. For such c, T_c is at least T_{n-3}, so

$$T_a + T_b \le 2n + (n-1) + (n-2) = 4n - 3.$$

Therefore $T_a \le 2n - \frac{3}{2}$. For $n \ge 21$, this implies $2a + 3 \le n$. This fact allows us to use Corollary 8.4, meaning that we have full use of the range $[a+1 \dots T_{n-1} - T_{a+1}]$.

Does $T_c - n$ fall into this range? For $n \ge 21$, the lower bound follows from

$$T_c - n \ge T_{n-3} - n \gg n > a + 1.$$

We prove the upper bound case by case.

Case 1. $c = n-3$. We can rearrange the upper bound condition

$$T_{n-3} - n = T_c - n \le T_{n-1} - T_{a+1}$$

to

$$T_a + a + 1 \le n + (n-1) + (n-2) = 3n - 3. \tag{6}$$

For n this large, $2a+3 \le n$ generously implies

$$a+1 \le n - \frac{3}{2},$$

which, together with the known $T_a \le 2n - \frac{3}{2}$, implies (6), so S exists and the Baron succeeds.

Case 2. $c = n-2$. Then $T_a + T_b = 3n-1$; therefore $T_a \le \frac{3n-1}{2}$. For the upper bound, we want

$$T_{n-2} - n = T_c - n \le T_{n-1} - T_{a+1},$$

which rearranges to

$$T_a + a + 1 \le n + (n-1).$$

Since we know $T_a \le \frac{3n-1}{2}$, it suffices that

$$a+1 \le \frac{n-1}{2},$$

which is mercifully equivalent to the already established condition $2a+3 \le n$. Therefore, the desired subset S exists and the Baron succeeds.

Case 3. $c = n-1$. Then $T_a + T_b = 2n$; therefore $T_a \le n$. If $b = a$, then $T_a = n$ and the Baron succeeds in one weighing. If $b = a+1$, then the Baron succeeds in two weighings by Lemma 7.2.

Now let us assume that $b \ge a+2$. Therefore, $T_b \ge T_{a+2} > T_a + (a+1) + (a+1)$. Therefore

$$\begin{aligned} 2n = T_a + T_b &> 2(T_a + (a+1)) \\ T_{a+1} &< n \\ 0 &< n - T_{a+1} \\ T_c &< n + T_{n-1} - T_{a+1} \\ T_c - n &< T_{n-1} - T_{a+1}, \end{aligned}$$

so $T_c - n$ fits in the desired range, the desired subset S exists, and the Baron succeeds.

The argument above proves that the Baron can convince his audience of the weight of the n-gram coin among n coins for $n \ge 37$. The theorem is completed by noting that Lemma 7.1 covers all smaller n.

9 Discussion

Is it surprising that the answer is two? That no matter how many coins there are, Baron Münchhausen can always prove the weight of one of them in just two weighings on the scale? It surprised us and it surprised most people we gave this puzzle to. At first, everyone expects that this sequence should tend to infinity, or at least grow without bound.

So we were most intrigued when we proved Theorem 3.1 and discovered that the problem always has a simple solution in three weighings. On reflection, however, maybe that discovery should have been less of a surprise. In standard coin-weighing puzzles, the person constructing the weighings is trying to find something out; so they are limited if nothing else by information-theoretic considerations, and as the number of coins involved increases, the problem usually becomes unequivocally more difficult. In this puzzle, however, Baron Münchhausen has complete information. So on the one hand, as the number of coins increases, the audience knows less and the Baron's task becomes more difficult; but on the other hand, the available resources for constructing interesting weighings also grow. In this case, it turns out that these forces balance to produce a bounded sequence.

In fact, demonstrating the weight of one coin among n in two weighings is *easy*. Think about what the Baron actually needs to do to satisfy the conditions outlined in the beginning of the proof, in Section 8. He must find some decomposition of $n + T_n$ into three triangular numbers, and find some subset of a certain collection of his coins that adds up to some number. The proof is long and hairy because we are trying to prove that this subset *always* exists, but the vast majority of the time this is trivial. How many ways are there to pick a subset of integers from fifty to a hundred, so that their sum will be three thousand? Or three thousand one? Gazillions![1] If the range one has to work with is reasonably large, and the target sum is comfortably between zero and the total sum of all the integers in one's range, then of course one can find a subset, or a hundred subsets.

Even more, any number n generally has *many* decompositions into a sum of three triangular numbers—on the order of the square root of n [5]. The proof in Section 8 is hairy also because we were proving that an *arbitrary* decomposition of $n + T_n$ into three triangular numbers leads to a solution, but in practice the Baron has the freedom to pick and choose among a great

[1] Yes, "gazillions" is a technical term in advanced combinatorics.

number of possible decompositions.

The ease of proving one coin in two weighings suggests two future directions. One can explore how many ways there are for Baron Münchhausen to prove himself right. One can also explore harder tasks that can be asked of him.

For the outermost example, we can ask the Baron to prove the weight of *all* the given coins. For $n = 6$ this task will match the following puzzle, authored by Sergey Tokarev [9], that appeared at the last round of the Moscow Math Olympiad in 1991:

> You have 6 coins weighing 1, 2, 3, 4, 5 and 6 grams that look the same, except for their labels. The number (1, 2, 3, 4, 5, 6) on the top of each coin should correspond to its weight. How can you determine whether all the numbers are correct, using the balance scale only twice?

This task is clearly harder, and indeed this sequence does tend to infinity: If the total number of coins is n, then the needed number of weighings is always greater than $\log_3 n$ [7]. And again, we give it as homework for the reader to prove this lower bound as well as the upper bound of $n-1$ weighings.

There is also a huge spectrum of possible intermediate tasks. For example, how many coins can the Baron show at once with at most two weighings? What is the smallest number of weighings the Baron needs to specify two coins? Or, given the total number of coins, how many weighings does the Baron need to show the weight of a particular coin? What if the audience can choose which coin's weight the Baron must prove? Which of these tasks can be done in a fixed maximum number of weighings, and which can not? What asymptotic behaviors of the number of needed weighings occur? What happens if we start using different families of sets of available coins, not just $1 \ldots n$? There is plenty to be curious about!

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