On quantum group $\mathbf{SL_q(2)}$.

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Introduction.

The goal of this paper is to analyze the notion of quantum group. There are two approaches to this notion:

In first approach, one describes a quantum group G in terms of a Hopf algebra $A = A(G)$ which plays the role of the algebra of functions on G. Then one studies the monoidal category of A-comodules, which is thought of as the category of representations of G. Our basic motivating example is the algebra A of regular functions on quantum group $SL_q(2)$ (see [R-T-F], [M]).

In second approach, one describes the quantum group in terms of a Hopf algebra $U \subset A^*$, which plays the role of universal enveloping algebra, and studies the tensor category of U-modules. This approach was initiated by Drinfeld [D] and Jimbo [J]. We use Lusztig's exposition (see [L]).

We decided to find a way from the first definition to the second one trying to understand what axiomatic and structures lied behind it.

We begin with the Hopf algebra $A = A_q$ of regular functions on $SL_q(2)$. This Hopf algebra supplies us with the material for axiomatic construction and generalizations. Then our axiomatic approach gives us the direction in which to make further investigation of the Hopf algebra of regular functions on $SL_q(2)$, and so on.

The results of this article could be formulated as follows:

1. We propose an axiomatic construction of Hopf algebras A and U, based on the observation that the quantum group G contains a quantum subgroup, which is just a usual Cartan subgroup. In this paper we give detailed analysis of $SL_q(2)$ case, starting from the one-dimensional torus and the root system of $SL(2)$.

2. Our construction leads us to a new class of quantum groups of $SL(2)$-type which seem to be the quantum analogues of metaplectic extensions of the group $SL(2)$.

3. Our approach forced us to introduce a notion of tetramodule (see [Kh]). We are thankful to V.Lyubashenko and S.Shnider who pointed out to us that this notion had already appeared in literature under many other names. For example: bidimodule [G-S], Hopf bimodule [L-S], two-sided two-cosided Hopf module [Sch], 4-module [S-S], bicovariant bimodule [W].

We plan to develop similar approach to other quantum groups in future publications.

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1. Algebra of regular functions on $\mathbf{SL_q(2)}$.

1.1. For every $q \in \mathbb{C}^*$ we consider the quantum group $SL_q(2)$. This group is defined by its algebra of functions A_q which is an algebra generated by four noncommuting elements (a, b, c, d), satisfying the following relations [R-T-F]:

$$(*) \qquad \begin{array}{cc} ab = q^{-1}ba & ac = q^{-1}ca \\ cd = q^{-1}dc & bd = q^{-1}db \\ bc = cb & ad - da = (q^{-1} - q)bc \end{array}$$

$$ad - q^{-1}bc = 1$$

Introduce matrices

$$Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat } (2, A_q)$$

.

$$Y^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & -1 \\ q^{-1} & 0 \end{pmatrix}$$

Then we can rewrite the relations $(*)$ in a more compact form:

$$\begin{aligned} YQY^t &= Q \\ Y^tQY &= Q \ . \end{aligned}$$

1.2. *Remark.* In a similar way one can describe the quantum group $GL_q(2)$. Namely, the relations

$$\begin{aligned} YQY^t &= x_1 Q \\ Y^tQY &= x_2 Q \end{aligned} \qquad x_1, x_2 \in \mathbb{C}^*$$

imply that $x_1 = x_2$. Hence, we can define the algebra of functions on the quantum group $GL_q(2)$ as the algebra generated by a, b, c, d, x with relations: 1) x is invertible element in the center; 2) $YQY^t = xQ = Y^tQY$.

1.3. The comultiplication in the algebra A_q is defined as follows:

$$(**)\qquad \begin{aligned} \Delta a &= a \otimes a + b \otimes c \\ \Delta b &= a \otimes b + b \otimes d \\ \Delta c &= c \otimes a + d \otimes c \\ \Delta d &= c \otimes b + d \otimes d \ . \end{aligned}$$

Using the natural imbeddings $i',\ i'' : A_q \to A_q \otimes A_q,\ \big(i'(x) = x \otimes 1,\ i''(x) = 1 \otimes x\big)$, we can rewrite comultiplication formulae $(**)$ as follows:

$$\Delta(Y) = i'(Y) \cdot i''(Y) \ ,$$

which is an equality in $\mathrm{Mat}(2, A_q \otimes A_q)$.

There exists an antipode i in the algebra A_q, and it is defined as follows:

$$\begin{aligned} a &\mapsto d \quad & b &\mapsto -q^{-1}b \\ d &\mapsto a \quad & c &\mapsto -qc \ . \end{aligned}$$

In a more compact form:

$$Y \mapsto (Q^{-1}YQ)^t.$$

1.4. Analyzing this algebra we note that it has the following important property:

Let $I \subset A_q$ be a two-sided ideal generated by b and c. Then I is a Hopf ideal in A_q, i.e. $\Delta I \in A \otimes I + I \otimes A$ and $i(I) \subset I$. The quotient Hopf algebra $S = A_q/I$ is isomorphic to the algebra of functions on the algebraic group $H = \mathbb{C}^*$:

$$S = \mathbb{C}[a, d]\big/(ad - 1)$$

$$\Delta a = a \otimes a \quad \Delta d = d \otimes d$$

Informally, this means that our quantum group $SL_q(2)$ contains the usual group $H = \mathbb{C}^*$ as a subgroup.

2. Axiomatic approach.

2.1. Let us fix a torus H (i.e. an algebraic group isomorphic to $\mathbb{C}^{*n}$) and denote by S the Hopf algebra of regular functions on H. We would like to study pairs (A, I) satisfying the following property:

Assumption I. A is a Hopf algebra (with multiplication m and comultiplication Δ) and $I \subset A$ is a two-sided Hopf ideal, such that A/I is isomorphic to S as a Hopf algebra.

2.2. Note that the comultiplication $\Delta : A \to A \otimes A$ leads to two S-comodule structures on A:

$$\begin{aligned} c_\ell &: A \to S \otimes A \qquad & c_\ell &= (pr \otimes id)\Delta \\ c_r &: A \to A \otimes S \qquad & c_r &= (id \otimes pr)\Delta \end{aligned}$$

where pr is the natural projection $A \to S = A/I$.

These structures define algebraic actions s_ℓ and s_r of the algebraic group H on A. These actions commute and preserve I.

We also consider the adjoint action of $H : ad_h = s_\ell(h) \cdot s_r^{-1}(h),\ h \in H$.

2.3. Consider an associated graded algebra $gr\, A$:

$$gr\, A = \bigoplus_{0 \leq n} gr_n\, A,$$

where

$$gr_n\, A = I^n / I^{n+1}\ .$$

It is easy to see that $gr\, A$ inherits the structure of a graded Hopf algebra with $gr_0 A$ equal to S. In particular, grA has the structure of a graded S-bicomodule.

2.4. We have two natural S-module structures on grA. These structures commute and preserve $gr_n A$.

The S-bicomodule and S-bimodule structures are compatible. We will describe, for example, the compatibility of the left S-comodule and right S-module structures: for any $h \in H, s \in S,\ x \in grA$

$$s_\ell(h)(xs) = s_\ell(h)(x) \cdot s_\ell(h)(s),$$

or, in other words,

$$c_\ell(xs) = c_\ell(x)\Delta s\ .$$

The other three relations are of the same type:

$$c_\ell(sx) = \Delta s \cdot c_\ell(x)$$

$$c_r(xs) = c_r(x)\Delta s$$

$$c_r(sx) = \Delta s \cdot c_r(x).$$

2.5. *Definition.* We call the linear space V an S-*tetramodule* if V is equipped with commuting left and right S-module structures, commuting left and right S-comodule structures, and the S-bimodule and S-bicomodule structures are compatible (see 2.4).

3. S-tetramodules.

3.1. For properties of tetramodules we use notations of [Kh]. See also [A], [S], bidimodules - [G-S], Hopf bimodules - [L-S], two-sided two-cosided Hopf modules - [Sch], 4-modules - [S-S], bicovariant bimodules - [W].

3.2. There is another shorter and more invariant way to describe the notion of tetramodule. Let us define a (0,1)-graded space to be a graded vector space $B = \oplus B_i$ such that $B_i = 0$ for $i \neq 0, 1$. In the category of (0,1)-graded spaces we can define the restricted tensor product by $U \otimes_r V = (U \otimes V)_0 \oplus (U \otimes V)_1 = U \otimes V/(U \otimes V)_2$.

Now define a restricted bialgebra as a (0,1)-graded vector space B together with morphisms $m : B \otimes_r B \to B$, $\Delta : B \to B \otimes_r B$, $\varepsilon : \mathbb{C} \to B$ and $\eta : B \to \mathbb{C}$, satisfying usual axioms of bialgebra.

Lemma. Let $B = (B_0 = S, B_1 = V)$ be a (0,1)-graded vector space. Then to define a structure of restricted bialgebra on B is the same as to define a structure of bialgebra on S and a structure of S-tetramodule on V.

3.3. Consider the case when S is the Hopf algebra of regular functions on a torus H. In this case we can give an explicit description of the category of S-tetramodules (for any S, see [A], [S]). We use the following standard

Lemma. Let W be an S-module equipped with the compatible algebraic action of the group H. Then $W = S \otimes W^H$, where W^H is the space of H-invariants.

3.4. Let us apply this lemma to our case. Let V be an S-tetramodule. Applying lemma 3.3 to the right action of H on V and the right multiplication by S we can write V as $V = V^H \otimes S$.

We want to describe an S-tetramodule structure on V in terms of some structures on the vector space V^H.

The right action of H on V^H is trivial. It is clear that V^H is ad_H-invariant, so the left action of H on V^H coincides with the ad_H action. Hence, knowing the ad_H action on V^H, we can reconstruct the left and right actions of H on V.

The right action of S on V is defined by decomposition $V = V^H \otimes S$. Now we have to reconstruct the left action of S on V.

Let Λ be the lattice of characters of H. Then $\Lambda \subset S$ is a basis of S. For $\lambda \in \Lambda$ consider operators $m_\ell(\lambda)$ and $m_r(\lambda)$ of left and right multiplications on λ in V, and set $L(\lambda) = m_\ell(\lambda)m_r(\lambda)^{-1}$. Then the axiom H3 implies that the operators $L(\lambda)$ commute with the right and the left action of H and hence preserve the subspace V^H.

So we have defined a homomorphism L of Λ into automorphisms of V^H, commuting with ad_H. Knowing L we can reconstruct the left action of S on V.

3.5. *Summary.* Let S be the Hopf algebra of regular functions on a torus H. Then the functor $V \to V^H$ gives an equivalence of the category of S-tetramodules with the category of algebraic H-modules equipped with the commuting action L of the lattice Λ .

4. Next step – $\mathbf{A/I^2}$.

4.1. Let us return to a Hopf algebra A with a Hopf ideal I, such that $gr_0 A$ equals S. We denote $gr_1 A$ by T. Then $T = I/I^2$ is an S-tetramodule. Let us denote T^H by M.

We assume that our tetramodule T satisfies the following assumption:

Assumption II. The space M is a direct sum of nontrivial nonequivalent one-dimensional representations of torus H.

4.2. *Example.* Let G be a reductive algebraic group, H its Cartan subgroup. Let $A = \mathbb{C}[G]$ be the Hopf algebra of regular functions on G and I the ideal of functions equal to 0 on H. Then $S = A/I$ equals $\mathbb{C}[H]$, $T = I/I^2$ is an S-tetramodule. The space $M = T^H$ is isomorphic to $(\mathcal{G}/\mathcal{H})^*$.

As an H-module, M is a direct sum of one-dimensional representations M_α which correspond to roots of G; in particular it satisfies Assumption II.

Consider a family of quantum deformations G_q of the group G. By this we mean a flat family of Hopf algebras A_q depending on some parameter q which for some value of q gives A. Let us assume that we also can flatly deform the ideal I such that the family of quotient Hopf algebras $H_q = A_q/I_q$ is constant and equals $S = \mathbb{C}[H]$. Under such deformations dimensions of different components of the space M_q can only drop, so it will satisfy Assumption II. If we take, for example, the Hopf algebra of functions on $SL_q(n)$ (see, for example [M2]) for generic q the space M will have only components M_α corresponding to roots α such that either α or $-\alpha$ is a simple root (see [Kh]).

4.3. Now we consider an S-tetramodule T satisfying the assumption II. Thus $M = \oplus M_\alpha$, where α runs some finite subset $\Sigma \subset \Lambda \setminus \{0\}$ and $\dim M_\alpha = 1$ for every α. The action L of the lattice Λ on M commutes with ad_H and hence preserves every subspace M_α. On the space M_α a homomorphism L (see 3.4) is given by a character $\gamma_\alpha : \Lambda \to \mathbb{C}^*$ that is by an element γ_α in H.

4.4. *Summary.* Under assumptions I and II an S-tetramodule T is fully described up to an isomorphism by a finite subset Σ in $\Lambda\backslash\{0\}$ and a map $\gamma : \Sigma \to H$.

4.5. Consider the adjoint action of the group H on the exact sequence

$$0 \to T \to A/I^2 \to S \to 0 \ .$$

Then the action on S is trivial and the action on T by assumption II does not have invariant vectors. Thus as an H-module this sequence canonically splits: $A/I^2 = S \oplus T$.

4.6. *Resume.* Under assumptions I and II an S-tetramodule T allows us to describe completely the structure of A/I^2 as an algebra and an S-bicomodule.

*. Passing to the dual picture.

*.1. General case.

***.1.1.** Let A be a Hopf algebra. We would like to describe the monoidal category Rep(A) of representations of A, i.e. monoidal category of left A-comodules. One of the standard ways to do it is to pass to modules over the dual algebra A^*. We define the multiplication in the vector space A^* to be dual to the opposite comultiplication in A: for $f, g \in A^*$ we define $f \cdot g\ (a)$ as $(g \otimes f)\ (\Delta(a))$.

The antipode can also be easily defined: $i(f)(a) = f(i(a))$.

Let $\rho : V \to A \otimes V$ be a left A-comodule. Then every functional $f \in A^*$ defines an operator $\rho(f) : V \to V$ as a through map $V \xrightarrow{\rho} A \otimes V \xrightarrow{f \otimes id} V$. It is clear that $\rho(f) \cdot \rho(g)$ equals $\rho(f \cdot g)$, i.e. a left A-comodule defines a left A^*-module (this is the reason why we prefer the multiplication in A^* to be *opposite* to the comultiplication in A). So we have described a fully faithful functor from $\mathrm{Rep}(A)$ into the category of left A^*-modules.

***.1.2.** If A is finite dimensional we define the comultiplication in A^* to be dual to the opposite multiplication in A:

$$(*). \qquad \Delta^* f(x \otimes y) = f(yx)$$

Then (A^*, m^*, Δ^*) is a Hopf algebra.

When A is infinite dimensional the formula $(*)$ defines $\Delta^* f$ as an element in $(A \otimes A)^*$ which is bigger than $A^* \otimes A^*$. Thus we can not consider A^* as a Hopf algebra.

There are two usual strategies how to deal with this difficulty. First is to choose a completion of $A^* \otimes A^*$ such that $\Delta^* f$ would be defined for every $f \in A^*$. Second is to choose a subalgebra $U \subset A^*$ on which Δ^* is defined, which is closed with respect to Δ^* and which is "big enough" (this means that U is dense in A^* in weak topology, or, equivalently, that the orthogonal complement of U in A is 0).

Our approach is to construct a Hopf subalgebra $U \subset A^*$ and describe some subcategory of U-modules which is close enough to the category $\mathrm{Rep}(A)$.

*.2. Basic example.

***.2.1.** We consider the following simple but very instructive case. Let $A = S = \mathbb{C}[H]$ be the Hopf algebra of regular functions on a torus H. Then the dual algebra S^* can be realized as the algebra of all functions on the lattice Λ of characters of S. There are several natural choices for a subalgebra $U \subset S^*$:

(i) Algebra $U = U^0$ - a free algebra generated by elements $\hat{h}$ ($h \in H \subset S^*$). This is a Hopf subalgebra, since $\Delta(\hat{h}) = \hat{h} \otimes \hat{h}$. Here we use $\hat{h}$ to differentiate the generator of an algebra from the point of a torus. Our example - $SL(2)$ - could be the most confusing since points of the torus are described by non zero complex numbers.

In fact, they often use even smaller subalgebra, generated by some subgroup of H. For example, in $SL(2)$ case we can take the smaller subalgebra generated by $\hat{q}, \hat{q}^{-1}$.

(ii) $U = U(\mathcal{H})$ - the enveloping algebra of the Lie algebra of H. This is a Hopf subalgebra.

(iii) $U = U^f$ - subalgebra generated by U^0 and $U(\mathcal{H})$. This subalgebra can be described as the algebra of all functionals, which are finite with respect to multiplicative action of S. This is a Hopf subalgebra.

(iv) $U = S^*$. In this case we have to complete $S^* \otimes S^*$ to $(S \otimes S)^*$ in order to be able to define Δ. We set $S^* \hat{\otimes} S^* := (S \otimes S)^*$. The algebra $S^* \hat{\otimes} S^*$ could be realized as the algebra of all functions on the lattice $\Lambda \oplus \Lambda$. If $f \in S^*$ then $\Delta f(\lambda_1, \lambda_2) = f(\lambda_1 + \lambda_2)$.

***.2.2.** Our main interest is the category of S-comodules = the category of algebraic representations of H. We can describe the category of algebraic representations of H as the category of U-modules, which are algebraic when restricted to H (when H is not a subalgebra of U we should say correspond to algebraic representation of H).

Thus, from our point of view, the dual object to S is any Hopf algebra U which is "dual" to S in the sense described above *together* with a category of U-modules which are algebraic when restricted to H.

***.2.3.** Later in this paper we prefer to take $U = S^*$ as this is the most general case; and it includes all other cases; or, alternatively one can take $U = U^0$ - in this case comultiplication formulae look clearer and simpler.

*.3. Axiomatic approach.

***.3.1.** Let A be a Hopf algebra with a Hopf ideal I, satisfying the assumption I.

We call a functional $f \in A^*$ I-finite if it vanishes on some power of the ideal I. The space of I-finite functionals we denote by U. This space is an algebra; and has a natural algebra filtration $U_0 \subset U_1 \subset \ldots$, where U_i consists of functionals which vanish on I^{i+1}. In particular, the subalgebra $U_0 = S^*$ contains H as a subgroup.

Definition. The (U, H)-module is a U-module such that its restriction to H is an algebraic representation of H. The category of (U, H)-modules we denote by $\mathcal{M}(U, H)$. As follows from the construction we have a canonical faithful functor $\mathrm{Rep}(A) \to \mathcal{M}(U, H)$. This functor is fully faithful provided U is dense in A, i.e. provided that the powers of the ideal I have 0 intersection.

***.3.2.** Let us describe in more detail the structure of the algebra U. We saw that $U_0 = S^*$. Also, it is clear that U is generated by U_1 as an algebra. In order to describe the structure of U_1 we suppose that A satisfies the assumption II. That means we can use decomposition $A/I^2 = S \oplus T$ from 4.5.

For every $\alpha \in \Sigma$, we fix a non-zero vector E_α in the one-dimensional space $M^*_{-\alpha}$. Since $T = \big(\bigoplus_{\alpha \in \Sigma} M_\alpha\big) \otimes S$, then E_α defines a morphism $S^* \to T^* \subset A^*$. In particular, it defines a family of elements $E_\alpha(f) \in T^*$ for $f \in S^*$. Now the S-bicomodule structure of T defines an S^*-module structure on the space of operators spanned by $E_\alpha(f)$. It is easy to describe this structure:

$$E_\alpha(f_1) f_2 = f_2(\alpha) E_\alpha(f_1 f_2)$$
$$f_2 E_\alpha(f_1) = E_\alpha(f_1 f_2) \ .$$

It is clear, that if for every α we fix $f_\alpha \in S^*$, then $U_1 = S^* \oplus \oplus_\alpha S^* E_\alpha(f_\alpha)$.

***.3.3.** Now we want to define the comultiplication on U. In order to do this, we need to complete $U \otimes U$. We set

$$U \hat{\otimes} U := (S^* \hat{\otimes} S^*) \otimes_{S^* \otimes S^*} (U \otimes U).$$

The comultiplication Δ on S^* is defined as in *.2.1.(iv). We also have

$$\Delta(E_\alpha(f)) = \Delta f \cdot (E_\alpha(1) \otimes \hat{\gamma}_{-\alpha} + 1 \otimes E_\alpha(1))$$

on U_1. Since $U \hat{\otimes} U$ is a subalgebra of $(A \otimes A)^*$, the multiplicativity of Δ and the fact that U is generated by U_1 implies that $\Delta U \in U \hat{\otimes} U$. Moreover, this gives us an explicit description of the comultiplication.

So the category $\mathcal{M}(U, H)$ becomes the tensor category; and we have a monoidal functor $\mathrm{Rep}(A) \to \mathcal{M}(U, H)$.

***.3.4.** *Resume.* We choose a subalgebra $U \subset A^*$ which is attentive to Hopf ideal I. Then we complete $U \otimes U$ in order to define Δ. This completion is easily described in terms of the completion $S^* \otimes S^*$ to $(S \otimes S)^*$. Then we restrict ourselves to the category $\mathcal{M}(U, H)$ which is our choice of dual object to A.

***.3.5** Later in this paper instead of U we will consider a smaller algebra $U^0 \subset U$, which is generated by $\hat{h}(h \in H)$ and by elements $E_\alpha(h)$ satisfying the relations:

$$\begin{aligned} E_\alpha(h_1)\hat{h}_2 &= \alpha(h_2)E_\alpha(h_1 h_2) \\ \hat{h}_2 E_\alpha(h_1) &= E_\alpha(h_1 h_2) \\ \Delta\hat{h} &= \hat{h} \otimes \hat{h} \\ \Delta(E_\alpha(h)) &= E_\alpha(h) \otimes \hat{h}\hat{\gamma}_{-\alpha} + \hat{h} \otimes E_\alpha(h) \ . \end{aligned}$$

The antipode in U^0 is defined as follows:

$$i(E_\alpha(h)) = -\hat{h}^{-1} E_\alpha(h) \hat{h}^{-1} \hat{\gamma}_{-\alpha}^{-1} \ .$$

*.4. Universal object.

***.4.1.** We denote by $\widetilde{U}$ a free algebra generated by elements $\hat{h}$ $(h \in H)$ and by elements $E_\alpha(h)$ satisfying the relations *.3.5.

We define a comultiplication Δ on $\widetilde{U}$ as in *.3.3. We define the category $\mathcal{M}(\widetilde{U}, H)$ of $(\widetilde{U}, H)$-modules as in *.3.1. Then we have an epimorphism of Hopf algebras $\widetilde{U} \to U$ which induces a fully faithful monoidal functor $\mathcal{M}(U, H) \to \mathcal{M}(\widetilde{U}, H)$; and hence, the functor $\mathrm{Rep}(A) \to \mathcal{M}(\widetilde{U}, H)$.

***.4.2.** *Summary.* We constructed a Hopf algebra $\widetilde{U}$ using only the S-tetramodule T. And for any S-tetramodule T we can construct a Hopf algebra $\widetilde{U}$.

*.5. $\mathbf{SL_q(2)}$.

***.5.1.** We apply our approach to the algebra A_q of functions on the quantum group $SL_q(2)$. The definition of A_q and $I \subset A_q$ is given in section 1.

The space M of right H-invariants in I/I^2 is two-dimensional: $M = M_\alpha \oplus M_{-\alpha}$, where α is a character of weight 2. And

$$\gamma_\alpha = q^{-1} \quad , \quad \gamma_{-\alpha} = q^{-1} \ .$$

***.5.2.** In this case the Hopf algebra $\widetilde{U}$ is generated by elements $\hat{h}, h \in H \approx \mathbb{C}^*$ and by elements $E(h) = E_\alpha(h)$ and $F(h) = E_{-\alpha}(h)$ satisfying the relations:

$$\begin{aligned} E(h_1)\hat{h}_2 &= h_2^{-2}E(h_1h_2) \\ \hat{h}_2E(h_1) &= E(h_1h_2) \\ F(h_1)\hat{h}_2 &= h_2^2F(h_1h_2) \\ \hat{h}_2F(h_1) &= F(h_1h_2) \end{aligned} \tag{1}$$

$$\begin{aligned} \Delta\hat{h} &= \hat{h}\otimes\hat{h} \\ \Delta E(h) &= E(h)\otimes(\widehat{q^{-1}}\hat{h}) + \hat{h}\otimes E(h) \\ \Delta F(h) &= F(h)\otimes(\widehat{q^{-1}}\hat{h}) + \hat{h}\otimes F(h) \ . \end{aligned} \tag{2}$$

***.5.3.** Let us see how the algebra $\widetilde{U}$ constructed from A_q corresponds to the definition of the enveloping algebra of quantum group $SL_q(2)$ (see [D],[J], we use notations of [L]).

We denote by K - an element in $\widetilde{U}$ which corresponds to $\hat{q} \in H,\ K = \hat{q} \in S^*$.

$$E = E(q),\ F = F(1) \ .$$

Then

$$\begin{aligned} KEK^{-1} = q^2E \qquad & KFK^{-1} = q^{-2}F \\ \Delta K &= K\otimes K \\ \Delta E &= K\otimes E + E\otimes 1 \\ \Delta F &= 1\otimes F + F\otimes K^{-1} \ . \end{aligned}$$

***.5.4.** We did not get the formula for $[E,F]$ in the quantum group $SL_q(2)$. We could not have done it, because we took into consideration in our axiomatic approach only the structure of $gr_0A_q \oplus gr_1A_q$. This structure forgets, for example, that $bc = cb$.

In the next sections we will develop our axiomatic approach to get a relation for $[E,F]$.

5. Universal objects.

5.1. Denote by $\mathcal{H}(S,T)$ - the category of graded Hopf algebras B, such that $B_0 = S, B_1 = T$; and B supplies T with the given S-tetramodule structure.

5.2. *Lemma.*

1) The category $\mathcal{H}(S,T)$ has the initial object B^i such that for any object B there exists a canonical morphism $B^i \to B$;

2) The category $\mathcal{H}(S,T)$ has the final object B^f such that for any object B there exists a canonical morphism $B \to B^f$;

3) The category $\mathcal{H}(S,T)$ has the minimal object B^m such that for any object B there exists a subalgebra $B' \subset B$ and a canonical epimorphism $B' \to B^m$.

Proof. It is easy to check that the construction in following paragraphs gives objects in question.

5.3. Given an S-tetramodule T we can consider T as an S-bimodule and construct a universal graded algebra $B^i(S,T)$, such that $B^i_0 = S$, and $B^i_1 = T$; and B^i supplies T with the given S-bimodule structure. B^i is a universal object as an algebra. Coalgebra structure is reconstructed on B^i by S-comodule structure on T and multiplicativity. The antipode is uniquely reconstructed on B^i by the antipode on S and its properties (see tensor algebra in [W]).

5.4. Now, given an S-tetramodule T we would like to construct a universal object with respect to coalgebra structure of T.

Given two S-bicomodules V_1 and V_2 we denote by $V_1 \otimes^S V_2$ a subspace in $V_1 \otimes V_2$ so that it is a kernel of an operator:

$$c_r \otimes id - id \otimes c_\ell : V_1 \otimes V_2 \to V_1 \otimes S \otimes V_2$$

(see [Kh], or cotensor product in [Sch]).

Given an S-bicomodule V denote $B^f_n(S,V)$ the space $(...(V \otimes^S V) \otimes ... \otimes^S V)$ (n times). It is easy to see that B^f_n is an S-bicomodule and is isomorphic to $B^f_k \otimes^S B^f_m$ for $k+m=n;\ k,m \geq 0$.

Denote $B^f(S,V) = \sum\limits_{0 \leq n} B^f_n$. The space B^f is supplied with the natural comultiplication structure. Namely, we define $\Delta B^f_n \to B^f \otimes^S B^f \subset B^f \otimes B^f$ so that for any $k,m \geq 0$ $k+m = n$ the composite map

$$\Delta B^f_n \to B^f \otimes^S B^f \overset{pr\otimes pr}{\to} B^f_k \otimes B^f_m$$

would be a canonical isomorphism.

Proposition. Above defined $B^f(S,V)$ is a universal graded coalgebra with $B^f_0 = S,\ B^f_1 = V$; and B^f supplies V with the given S-bicomodule structure.

5.5. *Statement.* Given an S-tetramodule T the universal coalgebra $B^f(S,T)$ is supplied with canonical Hopf algebra structure.

Proof. Actually, Hopf algebra B^f is universal in a stronger sense: namely, for any graded Hopf algebra C and compatible morphisms $C_0 \to S,\ C_1 \to T$; there is a canonical morphism $C \to B^f(S,T)$. It is easy to see that $B^f(S,T) \otimes B^f(S,T)$ is a universal free graded coalgebra $B^f(S \otimes S, S \otimes T + T \otimes S)$. We have natural morphisms: $m : S \otimes S \to S$

and $c_\ell + c_r : S \otimes T + T \otimes S \to T$. By universality property they define a multiplication map: $B^f(S,T) \otimes B^f(S,T) \to B^f(S,T)$.

The antipode could be always defined on T and by universality property reconstructed on B^f.

5.6. By definition of $B^i(B^f)$, we have a canonical morphism $\bar{\Delta}$:

$$\bar{\Delta} : B^i(S,T) \to B^f(S,T).$$

We put B^m equal to $\mathrm{Im}\bar{\Delta}$. For any object B we can canonically write $\bar{\Delta}$ as the composition map: $B^i \to B \to B^f$. This means that B has a subalgebra $B' = \mathrm{Im}B^i$, such that B^m is a surjective image of B'. End of proof of Lemma 5.2.

5.7. *Comments.* 1)The Hopf algebra B^f is in natural duality with the Hopf algebra $\widetilde{U}$ described in *.4. The morphism $\widetilde{U} \to U$ corresponds to a morphism $A \to \hat{B}^f$, where $\hat{B}^f$ is the natural completion of B^f (namely, $\hat{B}^f = \prod_i B_i^f$).

2) The construction of a minimal object B^m in the category $\mathcal{H}(S,T)$ is very similar to general construction of irreducible representations in representation theory. For example, in case of category $\mathcal{O}$ one describes explicitly basic modules M_λ and $\delta(M_\lambda)$; and then describes an irreducible object L_λ as the image of the canonical morphism $i : M_\lambda \to \delta(M_\lambda)$. Similar construction appears in Langlands classification of representations of real reductive groups (and, on more elementary level, in classification of representations of symmetric groups).

6. Back to $\mathbf{A_q}$.

6.1. In this and following sections we would like to describe the situation discussed above in $SL(2)$-case. Namely, consider

Assumption III. Our torus H is one-dimensional; and the space T^H of right H-invariants is the sum of two one-dimensional spaces of weights α and $-\alpha$.

6.2. Consider an object B in the category $\mathcal{H}(S,T)$. By definition $B_0 = S,\ B_1 = T$. We would like to discuss now what we can tell about B_2.

Consider the algebra $B^i(S,T)$. The space B_2^i would be isomorphic to $T \otimes_S T$. The space of right H-invariants in B_2^i is four-dimensional of weights $(-2\alpha, 0, 0, 2\alpha)$. The same is true for $B^f(S,T)$.

Consider the algebra $B = grA_q$, where A_q is the Hopf algebra of functions on $SL_q(2)$. Then $B_2 = I^2/I^3$. Its space of right H-invariants is three-dimensional and corresponds to three one-dimensional representations of H of weights $(-2\alpha,\ 0,\ 2\alpha)$. Hence, the canonical morphism $B^i \to grA_q$ has a kernel. This comes from the fact that in A_q (and, hence, in grA_q) we have the relation $bc = cb$ (see 1.1 and *.5.4); while in B^i this relation is absent.

6.3. We see that in $SL(2)$ case B^m is smaller than B^i and B^f ($\bar{\Delta}$ has a kernel). In fact, B^m is much smaller, since B^i has exponential growth – $\dim(B_k^i)^H = 2^k$, while

grA_q has polynomial growth – $\dim(gr_k A_q)^H = k+1$. This shows that it is important to investigate tetramodules for which $\bar{\Delta}$ has a nontrivial kernel.

6.4. To perform the calculations we would like to restrict ourselves to the study of the second graded component of $\bar{\Delta}(B^i \to B^f)$:

$$\bar{\Delta}_2 : T \otimes_S T \to T \otimes^S T.$$

The existence of the kernel gives us a hope to get a Hopf algebra similar to A_q.

6.5. So we come to a

Definition. We will call an S-tetramodule T an S-tetramodule of $SL(2)$-type if $S = \mathbb{C}[H]$, $(H = \mathbb{C}^*)$, $T^H = M_\alpha \oplus M_{-\alpha}$ and the operator $\bar{\Delta}$ on $(T \otimes_S T)^H$ has one-dimensional kernel of weight 0.

7. S-tetramodules of SL(2)-type.

7.1. Let T be an S-tetramodule of $SL(2)$-type. Let $s \in S$ be the coordinated function on $H \cong \mathbb{C}^*$. We have $T^H = M_\alpha \oplus M_{-\alpha}$. The weight α equals $n : \alpha = s^n$. Bimodule structure of T is described in 4.3. It is defined by two points of the torus γ_α and $\gamma_{-\alpha}$. In our case they correspond to two numbers $s(\gamma_\alpha)$ and $s(\gamma_{-\alpha})$ in $\mathbb{C}^*$. Let us choose t_1 and t_2 elements of M_α and $M_{-\alpha}$ respectively. We have

$$\begin{aligned}
\Delta t_1 &= s^n \otimes t_1 + t_1 \otimes 1 \\
\Delta t_2 &= s^{-n} \otimes t_2 + t_2 \otimes 1 \\
st_1 &= s(\gamma_\alpha) t_1 s \\
st_2 &= s(\gamma_{-\alpha}) t_2 \ .
\end{aligned}$$

The space of right H-invariants in $T \otimes_S T$ of weight 0 is spanned by $t_1 t_2$ and $t_2 t_1$. We apply $\bar{\Delta}$ to these elements:

$$\begin{aligned}
\bar{\Delta}(t_1 t_2) &= \alpha(\gamma_{-\alpha}) t_2 s^n \otimes t_1 + t_1 s^{-n} \otimes t_2 \\
\bar{\Delta}(t_2 t_1) &= t_2 s^n \otimes t_1 + (-\alpha)(\gamma_\alpha) t_1 s^{-n} \otimes t_2 \ .
\end{aligned}$$

7.2. Simple calculation immediately gives us the following

Lemma. T is an S-tetramodule of $SL(2)$-type iff $\alpha(\gamma_\alpha) = \alpha(\gamma_{-\alpha})$, or equivalently, $s(\gamma_\alpha)^n = s(\gamma_{-\alpha})^n$.

7.3. Let us denote $s(\gamma_{-\alpha})^{-1}$ by q. Then $s(\gamma_\alpha) = \varepsilon q^{-1}$, where ε is an n-th root of unity. For generic q the kernel of $\bar{\Delta}_2$ is spanned by the vector $t_1 t_2 - q^{-n} t_2 t_1$. It is easy to see that this vector generates a Hopf ideal in B^i. So we can take a quotient of B^i by this ideal.

We will denote the quotient algebra by $B_q(n, \varepsilon)$.

7.4. *Remark.* Understanding that the kernel could be of some importance, let us check whether $\bar{\Delta}\big|_{(T\otimes_S T)^H}$ has kernels of weights 2α or (-2α). It is easy to see that there exists a kernel of weight 2α (resp. -2α) iff $s(\gamma_\alpha)^n = -1$ (resp. $s(\gamma_{-\alpha})^n = -1$). Hence we can expect that the points q ($q^n = -1$) are special for $B_q(n, \varepsilon)$.

7.5. *Example.* $grA_q \approx B_q(2,1)$. Special points are $q = \pm i$.

*.6. Dual picture. Quantum commutator.

***.6.1.** We want to construct a Hopf algebra dual to $B_q(n, \varepsilon)$. In *.4 we have described an algebra $\widetilde{U}$ which is dual to $B^f(n, \varepsilon)$. The algebra $\widetilde{U}$ is generated by elements $\hat{h}, (h \in H)$ and by elements $E(h) = E_\alpha(h)$ and $F(h) = E_{-\alpha}(h)$ satisfying the relations:

(1)
$$\begin{aligned} E(h_1)\hat{h}_2 &= s(h_2)^{-n} E(h_1 h_2) \\ \hat{h}_2 E(h_1) &= E(h_1 h_2) \\ F(h_1)\hat{h}_2 &= s(h_2)^{n} F(h_1 h_2) \\ \hat{h}_2 F(h_1) &= F(h_1 h_2) \end{aligned}$$

(2)
$$\begin{aligned} \Delta\hat{h} &= \hat{h} \otimes \hat{h} \\ \Delta E(h) &= E(h) \otimes (\hat{q}^{-1}\hat{h}) + \hat{h} \otimes E(h) \\ \Delta F(h) &= F(h) \otimes (\hat{\varepsilon}\hat{q}^{-1}\hat{h}) + \hat{h} \otimes F(h) \end{aligned}$$

***.6.2.** To get a quotient algebra in $\widetilde{U}$ which is dual to $B_q(n, \varepsilon)$, we have to add the relation in $\widetilde{U}^2$ which is orthogonal to the image of $\bar{\Delta}_2$. That is

$$F(1)E(1) - q^{-n}E(1)F(1) = 0.$$

This relation is equivalent to the relation

$$s(h_2)^n E(h_1)F(h_2) - q^n s(h_1)^{-n} F(h_2)E(h_1) = 0$$

for any $h_1, h_2 \in H$.

***.6.3.** For any Hopf algebra B, which is an A-tetramodule we can define a quantum commutator on B with respect to A: $[\ ,\ ]_A$ (we would use this definition when B is graded Hopf algebra and $A = B_0$). This definition is due to Woronowicz (see external algebra in [W]). Namely, we have the standard adjoint action of A on B:

$$\mathrm{Ad}_A a : A \otimes B \to B \qquad b \mapsto \sum_k a_k^1 \cdot b \cdot i(a_k^2),$$

where $\Delta a = \sum_k a_k^1 \otimes a_k^2$.

Definition. We define $[\ ,\ ]_A$ as the difference of two operators from $B\otimes B$ to B:

$$B\otimes B \overset{id\otimes c_\ell}{\to} B\otimes A\otimes B \overset{S_{12}}{\to} A\otimes B\otimes B \overset{Ad_A\otimes id}{\to} B\otimes B \overset{m}{\to} B$$

$$B\otimes B \overset{c_r\otimes id}{\to} B\otimes A\otimes B \overset{id\otimes Ad_A}{\to} B\otimes B \overset{S_{12}}{\to} B\otimes B \overset{m}{\to} B.$$

Examples. 1) If B is a tetramodule over the field k, then $[b_1,b_2]_k = b_1b_2 - b_2b_1$.

2) If b_1 is a right A-coinvariant, and b_2 is a left A-coinvariant, then $[b_1,b_2]_A = b_1b_2 - b_2b_1$.

***.6.4.** We can write the relation in *.6.2 as a quantum commutator in $\widetilde{U}$ with respect to H (subalgebra generated by $\hat{h}$):

$$\alpha(h_2)E(h_1)F(h_2) - q^n(-\alpha)(h_1)F(h_2)E(h_1) = [E(h_1),F(h_2)]_H$$

If $s(h_1) = qs(h_2)^{-1}$, then the quantum commutator is proportional to the usual one: $[E(qh^{-1}),F(h)]_H = h^n[E(qh^{-1}),F(h)]$, in particular $[E(q),F(1)]_H = [E(q),F(1)]$. We have the following important relation:

$$\Delta[E(h_1),F(h_2)]_H = [E(h_1),F(h_2)]_H\otimes(\hat{\varepsilon}\hat{q}^{-2}\hat{h}_1\hat{h}_2) + (\hat{h}_1\hat{h}_2)\otimes[E(h_1),F(h_2)]_H,$$

which is equivalent to

$$\Delta[E(q),F(1)]_H = [E(q),F(1)]_H\otimes\hat{\varepsilon}\hat{q}^{-1} + \hat{q}\otimes[E(q),F(1)]_H.$$

***.6.5.** *Statement.* The algebra generated by elements $\hat{h}, E(h), F(h)\ \ (h\in H)$ and the relations (1), (2) and

$$[E(q),F(1)]_H = 0 \tag{3}$$

is a Hopf algebra.

This algebra is "dual" to $B_q(n,\varepsilon)$.

*.7. Metaplectic quantum groups of SL(2)-type.

***.7.1.** We have constructed the family of graded Hopf algebras $B_q(n,\varepsilon)$ which are analogues of grA_q. Our goal is to describe a family of quantum groups $G = G_q(n,\varepsilon)$ which we call metaplectic groups of $SL(2)$-type. The algebra $A_q(n,\varepsilon)$ of functions on this group has the property: $grA_q(n,\varepsilon) = B_q(n,\varepsilon)$. These algebras are the analogues of A_q. We will call $A_q(n,\varepsilon)$ the algebra of functions on metaplectic quantum group $SL_q(2)(n,\varepsilon)$ of $SL(2)$-type. In this section we construct a Hopf algebra $U_q(n,\varepsilon)$ which is dual to $A_q(n,\varepsilon)$ and which could be considered as the enveloping algebra of metaplectic quantum group $SL_q(2)(n,\varepsilon)$.

***.7.2.** Using the decomposition $A/I^2 = S \oplus T$ (see 4.5), we see that $U_(n,\varepsilon)^1 = S^* \oplus T^*$. Thus, we have a canonical morphism of Hopf algebras $\widetilde{U}_q(n,\varepsilon) \to U_q(n,\varepsilon)$. Passing to associated graded algebras we have a morphism ϕ: $\widetilde{U} = gr\widetilde{U} \to grU = \oplus B_i^*$. As we saw this morphism gives us an isomorphism: $grU = \widetilde{U}/K$, where K is the relation $[E(q), F(1)] = 0$. This implies that $U = \widetilde{U}/K'$, where K' is the relation $[E(q), F(1)] + Q = 0$, where $Q \in \widetilde{U}^1$. Since ϕ is equivariant with respect to adjoint action of H and K is invariant with respect to this action, this implies that Q is ad-invariant and hence $Q \in \widetilde{U}_0 = S^*$. Since ϕ is a morphism of Hopf algebras, then Q satisfies the relation (see *.6.4):

$$\Delta Q = \hat{q} \otimes Q + Q \otimes \hat{\varepsilon}\hat{q}^{-1}. \qquad (*)$$

***.7.3.** *Statement.* All solutions of $(*)$ are proportional to

$$\hat{q} - \hat{\varepsilon}\hat{q}^{-1}.$$

We normalize Q by the condition $Q(s) = 1$, so we will choose

$$Q = \frac{\hat{q} - \hat{\varepsilon}\hat{q}^{-1}}{q - \varepsilon q^{-1}}.$$

***.7.4.** *Theorem.* The elements $\hat{h}, E(h), F(h)$ satisfying the relations *.6.1 and the relation

$$[E(q), F(1)]_H = \frac{\hat{q} - \hat{\varepsilon}\hat{q}^{-1}}{q - \varepsilon q^{-1}} \qquad (3)$$

generate a Hopf algebra.

This is the announced Hopf algebra $U_q(n,\varepsilon)$.

***.7.5.** We denote $E(q)$ by E, $F(1)$ by F. We can rewrite the definition of metaplectic quantum groups in the more compact form:

Definition. The Hopf algebra $U_q(n,\varepsilon)$ is generated by multiplicative elements $\hat{h}$ ($h \in H$), and E and F satisfying the relations:

$$\begin{aligned} \hat{h}E\hat{h}^{-1} &= s(h)^n E \\ \hat{h}F\hat{h}^{-1} &= s(h)^{-n} F \end{aligned} \qquad (1)$$

$$\begin{aligned} \Delta\hat{h} &= \hat{h} \otimes \hat{h} \\ \Delta E &= E \otimes 1 + \hat{q} \otimes E \\ \Delta F &= F \otimes \hat{\varepsilon}\hat{q}^{-1} + 1 \otimes F \end{aligned} \qquad (2)$$

$$[E,F] = \frac{\hat{q}-\hat{\varepsilon}\hat{q}^{-1}}{q-\varepsilon q^{-1}} \tag{3}$$

***.7.6.** *Examples.* 1.The subalgebra of $U_q(2,1)$, generated by $E, F, K = \hat{q}$ gives us the usual definition of universal enveloping algebra of quantum group $SL_q(2)$ (see [L]). We think that our approach has some advantages. For example, we have only one one-dimensional representation of our algebra – the trivial one.

2.The subalgebra of $U_q(1,1)$ generated by the $E, F, K = \hat{q}$ gives us the usual definition of universal enveloping algebra of quantum group $PSL_q(2)$.

3. The Hopf algebra $U_q(2,-1)$ could be described as the specialization of $GL_{p,q}(2)$ (see [M2]), when $p = -q$.

8. Hopf algebra of regular functions on metaplectic quantum groups of SL(2)-type.

8.1. *Axiomatic approach.* We would like now to describe the algebra $A_q(n,\varepsilon)$ of functions on the metaplectic group $G_q(n,\varepsilon)$. This should be a Hopf algebra dual to $U_q(n,\varepsilon)$ such that $grA_q(n,\varepsilon) = B_q(n,\varepsilon)$.

In general we are not sure whether in all cases it is possible to construct an algebra A which is a Hopf algebra in a standard sense. The reason is that the only things we really can describe with our approach are the quotients A/I^n for all n. This means that the algebra which we can construct is the completion of A with respect to powers of ideal I. In other words, the natural dual object to U will be an algebra A with a two-sided ideal I complete with respect to the I-adic topology defined by powers I^n of the ideal I.

What about comultiplication. If we want it to be a morphism $\Delta : A \to A \otimes A$, then we do not know how to construct it (in fact there is now reason to expect that the comultiplication will be defined on the whole completed algebra A). However, working in this situation it is natural to replace $A \otimes A$ by a completed tensor product $A\hat{\otimes}A$. Namely, consider a two-side ideal $I' = I \otimes A + A \otimes I$ of the algebra $A \otimes A$ and denote by $A\hat{\otimes}A$ the completion of $A \otimes A$ in I'-adic topology.

8.2. It turns out, that with this definition we can define on A the natural structure of a completed Hopf algebra. In other words, we will construct a pair (A, I), where I is a two-sided ideal of A and A is completed in I-adic topology, and a comultiplication $\Delta : A \to A\hat{\otimes}A$, which satisfies all the axioms of Hopf algebra.

Namely, consider the completed algebra $\hat{B}^f$ corresponding to our tetramodule $T = T(n,\varepsilon)$. It is easy to see that it is a completed Hopf algebra in a sense described above, and that it is in natural duality with the algebra $\widetilde{U}(n,\varepsilon)$. Now let us denote by K' the kernel of the natural projection $\widetilde{U}(n,\varepsilon) \to U(n,\varepsilon)$ described in *.7.2 and define $A(n,\varepsilon) \subset \hat{B}^f$ to be the orthogonal complement of K', i.e. $A = \{b \in B | (b, K') = 0\}$.

Since $\widetilde{U}(n,\varepsilon) \to U(n,\varepsilon)$ is a morphism of Hopf algebras, A is a completed Hopf algebra dual to $U(n,\varepsilon)$.

*.8. Representations of $\mathbf{U_q(n,\varepsilon)}$.

***.8.1.** Let $U = U_q(n,\varepsilon)$ be the universal enveloping algebra of metaplectic group $G_q(n,\varepsilon)$. We denote by $\mathcal{M}(G)$ the category $\mathcal{M}(U,H)$ of (U,H)-modules. We consider the subcategories $\mathcal{F}(G) \subset \mathcal{O}(G) \subset \mathcal{M}(G)$, where $\mathcal{F}$ is the category of finite dimensional modules and $\mathcal{O}$ is the category of $\mathcal{O}$-modules, i.e. finitely generated (U,H)-modules M for which weight spaces M_i are 0 for large i.

The comultiplication in U allows to define a tensor product of (U,H)-modules and hence defines on each of this categories the structure of monoidal category.

***.8.2.** Let l be a divisor of n. Then the group H has a unique cyclic subgroup C of order l. It is clear from our definitions that this subgroup lies in the center of the algebra $U(n,\varepsilon)$. In other words, C can be considered as the central subgroup in the metaplectic quantum group $G_q(n,\varepsilon)$.

We can consider a quotient quantum group G/C. Its universal enveloping algebra $U(G/C)$ equals U/I,where I is the ideal generated by $\{\hat{c}-1|c\in C\}$. The corresponding algebra of functions $A(G/C)$ can be described as a subalgebra of C-invariant elements in $A(G)$ with respect to the natural left (equal to right) action of C on $A(G)$. It is easy to see that $U(G/C)$ is the algebra of type $U(n/l,\varepsilon^l)$. In particular, we have the natural monoidal functor $\mathcal{M}(G/C)\to\mathcal{M}(G)$, which transforms $\mathcal{O}$-modules into $\mathcal{O}$-modules and finite dimensional modules into finite dimensional ones.

Examples. 1) Let $l=n$. Then G/C is isomorphic to $PSL_q(2)$.

2) Let n be even, $l=n/2$. If $\varepsilon^l=1$ then G/C is isomorphic to $SL_q(2)$.

***.8.3.** Let l be the minimal positive multiple of $n/2$ such that $\varepsilon^l=1$, that is $l=n$ if n is odd, or n is even and $\varepsilon^{n/2}=-1$; $l=n/2$ if n is even and $\varepsilon^{n/2}=1$.

Proposition. Let C be the central subgroup, corresponding to l. Then for generic q any finite dimensional representation is trivial on the subgroup C. In other words, the functor: $\mathcal{F}(G)\to\mathcal{F}(G/C)$ is an equivalence of the categories.

Thus, the monoidal category of finite dimensional representations of a metaplectic group G is equivalent to the category of finite dimensional representations of one of the classical groups $SL_q(2)$ or $PSL_q(2)$.

Proof. Let V be a finite dimensional (U,H)-module. We can assume that it is irreducible. Standard argument shows that V has a highest weight vector v_0, such that v_0 has the weight N, and $Ev_0=0$. Let us denote F^iv_0 by v_i. The weight of v_i equals $N-ni$. The commutation relation for E and F shows that

$$Ev_i = [i]_{q^{n/2}}(q^{N-n(i-1)/2}-\varepsilon^N q^{-N+n(i-1)/2})/(q-\varepsilon q^{-1}),$$

where $[i]_a=(q^a-q^{-a})/(q-q^{-1})$.

There exists $k\neq 0$ such that $v_k\neq 0$ and $Fv_k=0$. Then the vectors $v_0,\ldots,v_k$ generate $k+1$-dimensional irreducible representation of G. Thus for generic q we have that $2N=nk$ and $\varepsilon^N=1$. Hence $N\in l\mathbb{Z}$.

***.8.4.** Let $C \subset H$ be a central subgroup in G. For any character ξ of C we denote by $\mathcal{M}_\xi(G)$ the subcategory of modules on which C acts via character ξ. Clearly $\mathcal{M}(G) = \oplus_\xi \mathcal{M}_\xi(G)$ and $\mathcal{M}_\xi \otimes \mathcal{M}_\eta \subset \mathcal{M}_{\xi\eta}$. Similarly for $\mathcal{O}(G)$ and $\mathcal{F}(G)$.

***.8.5.** Fix a subgroup C as in *.8.3. Then the category $\mathcal{F}(G)$ is one of two classical categories. For any character ξ we have compatible right and left actions of this classical category $\mathcal{F}(G) = \mathcal{F}(G/C)$ on the category $\mathcal{M}_\xi(G)$ and, in particular, on the category $\mathcal{O}_\xi(G)$.

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