ON QUANTUM GROUP $GL_{p,q}(2)$

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Abstract. In the Hopf algebra $Gl_{p,q}(2)$ the determinant is central iff $p = q$. In this case we put determinant to be equal to 1 to get $SL_q(2)$. In this paper we consider the case when p/q is a root of unity; and, consequently, a power of the determinant is central.

Introduction.

In the paper [B-Kh] the Universal enveloping algebras for metaplectic quantum groups of $SL(2)$-type were constructed. As we look back at the history of $SL_q(2)$ we can notice that on the levels of formulas the algebra of functions on $SL_q(2)$ was always presented the same way ([M], [R-T-F]), while the Universal enveloping algebra had different presentations. This fact encouraged me and Joseph Bernstein [B-Kh] to consider the enveloping algebra as the secondary object in comparison with the Hopf algebra of functions. As a byproduct of our axiomatic search we constructed the metaplectic quantum groups of $SL(2)$-type. This construction was carried out on the level of universal algebras.

On my way to find nice formulas for algebra of functions on metaplectic quantum group, I found some nice formulas for other algebras. These formulas are presented in this paper.

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1. Algebra of regular functions on $GL_{p,q}(2)$.

1.1. For every $p, q \in \mathbb{C}^*$ we consider the bialgebra $Mat_{p,q}$ which is generated by four noncommuting elements (a, b, c, d), satisfying the following relations (see, [M], [O-W] and references there):

$$\begin{aligned} ab &= p^{-1}ba & ac &= q^{-1}ca \\ cd &= p^{-1}dc & bd &= q^{-1}db \\ bc &= q^{-1}pcb & ad - da &= (p^{-1} - q)bc. \end{aligned} \qquad (*)$$

Introduce matrices

$$Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mat\ (2, Mat_{p,q})$$

$$P = \begin{pmatrix} 0 & -1 \\ p^{-1} & 0 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & -1 \\ q^{-1} & 0 \end{pmatrix}.$$

Then we can rewrite the relations $(*)$ in a more compact form:

$$\begin{aligned} YPY^t &= DP \\ Y^tQY &= DQ, \end{aligned}$$

where D is an element in $Mat_{p,q}$ and has a meaning of quantum determinant.

Comment. The matrix P defines a quantum plane as an algebra generated by two generators x, y and the relation $xy = p^{-1}yx$. Analogously the matrix Q defines a quantum plane. We consider $Mat_{p,q}$ as an algebra of operators preserving quantum plane defined by P and dual quantum plane defined by Q.

1.2. The comultiplication in the algebra $Mat_{p,q}$ is defined as follows:

$$\begin{aligned} \Delta a &= a \otimes a + b \otimes c \\ \Delta b &= a \otimes b + b \otimes d \\ \Delta c &= c \otimes a + d \otimes c \\ \Delta d &= c \otimes b + d \otimes d\ . \end{aligned} \tag{**}$$

Using the natural imbeddings $i',\ i'' : Mat_{p,q} \rightarrow Mat_{p,q} \otimes Mat_{p,q}$, $\big(i'(x) = x \otimes 1,\ i''(x) = 1 \otimes x\big)$, we can rewrite comultiplication formulae $(**)$ as follows:

$$\Delta(Y) = i'(Y) \cdot i''(Y)\ ,$$

which is an equality in $Mat(2, Mat_{p,q} \otimes Mat_{p,q})$.

1.3. This algebra has a multiplicative quantum determinant $D = det_{p,q}(Y)$ (see [M], [O-W], [K]):

$$D = da - pcb = da - qbc = ad - p^{-1}bc = ad - q^{-1}cb.$$

Multiplicativity means that $\Delta D = D \otimes D$, or, equivalently, $det_{p,q}(Y_1Y_2) = det_{p,q}(Y_1) \cdot det_{p,q}(Y_2)$ whenever the entries of Y_1 commute with the entries of Y_2.

Localizing by D^{-1} we will get the algebra of functions on $GL_{p,q}(2)$. This localization can be described easily due to the fact that D is normalizing (see [M]):

$$\begin{aligned} Da &= aD \qquad Db = p^{-1}qbD \\ Dc &= pq^{-1}cD \qquad Dd = dD. \end{aligned}$$

1.4. The bialgebra $Fun(GL_{p,q})$ is the Hopf algebra; an antipode S is defined by: $S(Y) = Y^{-1}$. Specifically:

$$\begin{aligned} S(a) &= dD^{-1} \qquad S(b) = -pbD^{-1} \\ S(c) &= -p^{-1}cD^{-1} \qquad S(d) = aD^{-1}. \end{aligned}$$

In a more compact form:

$$Y^{-1} = S(Y) = PY^tP^{-1}D^{-1} = D^{-1}Q^{-1}Y^tQ.$$

1.5. If $p = q$ then D is central; and we can take quotient of $Fun(GL_{p,q}(2))$ by the Hopf ideal generated by $D - 1$. We would get standard $Fun(SL_q(2))$ (see [Kas]).

1.6. Suppose now that $p^{-1}q$ equals ξ, where ξ is the n-th root of unity: $\xi^n = 1$. In this case D^n is central. An ideal generated by $D^n - 1$ is a Hopf ideal, so we can consider a quotient algebra, which we would denote by $Fun(SL_{q,\xi}(2))$:

$$Fun(SL_{q,\xi}(2)) = Fun(GL_{p,q}(2))/(D^n - 1).$$

2. Axiomatic approach.

2.1. Analyzing the Hopf algebra $A = Fun(SL_{q,\xi}(2))$ we note that it has the following important property:

Let $I \subset A$ be a two-sided ideal generated by b and c. Then I is a Hopf ideal in A, i.e. $\Delta I \in A \otimes I + I \otimes A$ and $S(I) \subset I$. The quotient Hopf algebra A/I is isomorphic to the algebra of functions on the algebraic group $\mathbb{C}^* \otimes \mathbb{Z}_n$:

$$\mathbb{C}[a, d, D, D^{-1}]/(ad - D, D^n - 1)$$

$$\Delta a = a \otimes a \quad \Delta d = d \otimes d \quad \Delta D = D \otimes D$$

$$S(a) = dD^{-1} \quad S(d) = aD^{-1} \quad S(D) = D^{-1}.$$

Or, equivalently:

$$\mathbb{C}[a, a^{-1}, D, D^{-1}]/(D^n - 1)$$

$$\Delta a = a \otimes a \qquad \Delta D = D \otimes D$$

$$S(a) = a^{-1} \quad S(D) = D^{-1}.$$

Informally, this means that our quantum group $A = Fun(SL_{q,\xi}(2))$ contains the group $\mathbb{C}^* \otimes \mathbb{Z}_n$ as a subgroup.

3. Dual picture.

3.1. Let us denote by T^2 the two-dimensional torus. Denote by t a point of T^2. To each point $t = (x_1, x_2)$ we can correspond a generator $\hat{t}$ in the group algebra of torus. This correspondence is multiplicative — $\widehat{t_1 t_2} = \hat{t}_1 \hat{t}_2$.

We can describe (we use notations similar to [B-Kh]) a universal enveloping algebra $U_{p,q}(2)$ of $GL_{p,q}(2)$ as a Hopf algebra generated by $\hat{t}$ ($t \in T^2$) and two elements E, F, satisfying the relations:

$$\begin{aligned} \hat{t}E\hat{t}^{-1} &= \alpha(t)E \\ \hat{t}F\hat{t}^{-1} &= (-\alpha)(t)F \end{aligned} \tag{1}$$

$$\Delta\hat{t} = \hat{t}\otimes\hat{t}$$
$$\Delta E = E\otimes 1 + Q_1\otimes E \qquad (2)$$
$$\Delta F = F\otimes Q_2^{-1} + 1\otimes F$$

(3) $$[E,F] = \frac{Q_1 - Q_2^{-1}}{q - p^{-1}}.$$

Here Q_1, Q_2 are generators corresponding to points of the torus with coordinates (q,p^{-1}) and (p,q^{-1}) respectively; and α denotes the weight $(1,-1)$ on torus: if $t=(x,y)$, then $\alpha(t)=xy^{-1}$.

Comments. 1. The commutator $[E,F]$ is the element X of an algebra, generated by torus, satisfying the equation: $\Delta X = Q_1\otimes X + X\otimes Q_2^{-1}$. In denominator (3) we can choose any constant. We chose our constant so that the evaluation of the right hand side element on the weight $(1,0)$ would be equal to 1.

2. In this case and in the following cases, the antipode is uniquely defined and could be easily recovered:

$$S(\hat{t}) = \hat{t}^{-1} \quad S(E) = -Q_1^{-1}E \quad S(F) = -Q_2F.$$

3.2. To get the universal algebra $U_q(2)$ of $SL_q(2)$ we have to put $p=q$ and to take the group subalgebra of $T^1\subset T^2$ generated by elements $\hat{h}$ corresponding to $(h,h^{-1})\in T^2$. Denote by K an element corresponding to (q,q^{-1}), then we would have the following relations:

(1) $$\hat{h}E\hat{h}^{-1} = h^2E$$
$$\hat{h}F\hat{h}^{-1} = h^{-2}F$$

(2) $$\Delta\hat{h} = \hat{h}\otimes\hat{h}$$
$$\Delta E = E\otimes 1 + K\otimes E$$
$$\Delta F = F\otimes K^{-1} + 1\otimes F$$

(3) $$[E,F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

3.3. Consider the case $SL_{q,\xi}(2)$: namely, $p^{-1}q=\xi$, where $\xi^n=1$. We can describe the universal enveloping algebra $U_{q,\xi}(2)$ as a subalgebra in $U_{p,q}(2)$, generated by elements $E,\ F,\ \hat{h}=(h,h^{-1})\in T^2$ for $h\in T$ and $W=(1,\xi)\in T^2$ $(W^n=1)$.

(1) $$WEW^{-1} = \xi^{-1}E \qquad \hat{h}E\hat{h}^{-1} = h^2E$$
$$WFW^{-1} = \xi F \qquad \hat{h}F\hat{h}^{-1} = h^{-2}F$$

$$\Delta W = W\otimes W$$
(2) $$\Delta\hat{h} = \hat{h}\otimes\hat{h}$$
$$\Delta E = E\otimes 1 + W\hat{q}\otimes E$$
$$\Delta F = F\otimes W\hat{\xi}\hat{q}^{-1} + 1\otimes F$$

$$(3) \qquad [E,F] = \frac{\hat{q} - \hat{\xi}\hat{q}^{-1}}{q - \xi q^{-1}} W.$$

References

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