Dragons and Kasha

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Abstract

Kasha-eating dragons introduce representation theory.

This is how my ex-husband Joseph Bernstein used to start his course in representation theory.

> Suppose there is a four-armed dragon on every face of a cube. Each dragon has a bowl of kasha in front of him. Dragons are very greedy, so instead of eating their own kasha, they try to steal kasha from their neighbors. Every minute every dragon extends four arms to the four neighboring cube's faces and tries to get the kasha from the bowls there. As four arms are fighting for every bowl of kasha, each arm manages to steal one-fourth of what is in the bowl. Thus each dragon steals one-fourth of the kasha of each of his neighbors, while all of his own kasha is stolen, too. Given the initial amounts of kasha in every bowl, what is the asymptotic behavior of the amounts of kasha?

Why do these dragons eat kasha? Kasha is very healthy. But for mathematicians, kasha represents a continuous entity. You can view the amount of kasha in a bowl as a real number. Another common food that works for this purpose is soup, but liquid soup is difficult to steal with your bare hands. We do not want to see soup spilled over our cube, do we?

How does this relate to representation theory? For starters, it relates to linear algebra. We can consider the amounts of kasha as six real numbers, as there are six bowls, one on each of the six faces of the cube. We can view this six-tuple that represents kasha at each moment as a vector in a six-dimensional vector space of possible amounts of kasha. To be able to view the amounts of kasha as a vector, we need to make a leap of faith and assume that negative amounts of kasha are possible. I just hope that if my readers have enough imagination to envision six four-armed dragons on the faces of the cube, then they can also imagine negative kasha. It is not that difficult to imagine. The bowl with -2 pounds of kasha means that if you put two pounds of kasha into this bowl, it becomes empty. For those who wonder why dragons would fight for the negative kasha, this is how mathematics works. We make unrealistic assumptions, solve the problem, and then hope that the solution translates to reality anyway.

Actually, we need to make an even bigger leap of faith and assume that the amounts of kasha are complex numbers. If you can imagine negative kasha, you ought to be able to imagine an imaginary kasha.

Back to the dragons. After all the kasha is redistributed as a consequence of four arms fighting and stealing, the result is a linear operator acting on our vector space, which we will call the *stealing operator*.

To answer the question in the puzzle we can calculate the eigenvalues and eigenvectors of the stealing operator, use them to diagonalize the operator matrix, which will allow us to calculate its powers and find out the asymptotic behavior. But where is the beauty? And what does it have to do with representation theory? An advanced mathematician might know the answer to the puzzle without calculation. S/he might notice that the stealing operator is a Markov matrix that has a steady state: if all bowls have the same amounts of kasha, the arm-fighting and stealing is a waste of time, nothing changes. Then s/he might be able to explain why this process is ergodic and thus the asymptotic behavior is the steady state.

But this essay is an ode to representation theory, so let us abandon Markov and find a group: representation theory always wants a group. We will use the group of rigid motions of the cube. The group acts on the six-tuples of the amounts of kasha. This action is called a representation of the group.

Our dragons respect the group action. Each dragon on each face does exactly the same thing. In other words, the stealing operator commutes with any motion of the cube: you can swap stealing kasha with rotating the cube. If dragons steal kasha first and then the cube is rotated, the result is the same as it would be if they had done these actions in the opposite order. An operator that commutes with the action of the group on our vector space is called an *intertwining operator* of this representation. That means our stealing operator is actually an intertwining operator.

Now we are well into representation theory and we want to use the following corollary of the powerful and beautiful Schur's Lemma.

Corollary 1. *If a complex representation of a group can be decomposed into non-isomorphic irreducible representations, then the intertwining operator acts as a scalar on each irreducible representation of the group.*

That means it is a good idea to decompose our six-dimensional vector space of the amounts of kasha into smaller subspaces that are not changed by movements of the cube.

The cube has a natural mirror symmetry that swaps the amounts of kasha on the opposite sides of the cube. That means we can decompose the space into two 3-dimensional subspaces that do not change after any rotation: the first subspace has the same amounts of kasha on the opposite sides, and the second subspace has the opposite amounts of kasha on the opposite sides.

We can also notice that if everyone has the same amount of kasha, then it stays the same after a movement of the cube.

Thus we can decompose the vector space into the following three representations:

- One-dimensional. Every dragon has the same amounts of kasha.

- Two-dimensional. Dragons on the opposite sides have the same amounts of kasha and the total amount of kasha is zero.
- Three-dimensional. Dragons on the opposite sides have the opposite amounts of kasha.

If these representations are irreducible, then they have to be non-isomorphic as they have different dimensions. In this case the stealing operator will act like a multiplication by a scalar. Even if these representations are not irreducible, the stealing operator can still act as a scalar. In any case, now we need to calculate how the stealing operator acts on each representation.

- Every dragon has the same amounts of kasha. The stealing operator acts as identity.
- Dragons on the opposite sides have the same amounts of kasha and the total amount of kasha is zero. The neighbors of one dragon have the opposite amounts of kasha as he and his opposite dragon have together. That means the neighbors have -2 times the amount of kasha the dragon has. The stealing operator acts as multiplying by $-1/2$.
- Dragons on the opposite sides have the opposite amounts of kasha. Each dragon is stealing from two pairs of dragons that are opposite each other. The total of the kasha of the neighbors of one dragon is zero. The stealing operator acts as zero.

Now we know exactly what happens each time, and we see that asymptotically every dragon will have the same amount of kasha. And to tell you a secret, these representations are indeed irreducible.

Now let's use this method to solve a similar problem, where there are n dragons sitting on the sides of an n-gon. Each dragon has two arms and steals half of the kasha from his neighbors. Hey, wait a minute! Why do we call them dragons? We do not need to do that. We can assume that these are greedy people with bad manners. But the question is still the same: What is the asymptotic behavior of the amounts of kasha?

There is a steady state with all the kasha the same. So you might expect that the amounts converge to this state. Then, why would I offer this puzzle?

Let's see what happens. The group we can use here is the rotation group of the n-gon. This group is commutative. Now we can use another corollary to Schur's Lemma:

Corollary 2. *All irreducible complex representations of an Abelian group are one-dimensional.*

Let's pick a designated person with bad manners, and call him person 1. We can assume that the amount of kasha he has is 1. Suppose his neighbor to the right has w kasha. If this is an irreducible representation, then rotating by one person to the right multiplies the amounts of kasha by a scalar which must be w. After we rotate n times, where n is the total number of people, we get back to person 1 while each rotation multiplies by w, so $w^n = 1$. Thus, w is a root of unity and for each such root we have an irreducible representation.

How does the stealing operator act on this one-dimensional representation? Our designated person has w kasha on the right and w^{-1} on the left. So after the first round he will have $(w + w^{-1})/2$ kasha. This is our multiplication coefficient. Given that $w = e^{2\pi ik/n}$, for $0 \le k < n$, we get that the kasha multiplies by $(e^{2\pi ik/n} + e^{-2\pi ik/n})/2 = \cos 2\pi k/n$. We are interested in the asymptotic behavior, which is non-trivial when the absolute value of the cosine is 1. This can only happen in two cases. For $k = 0$, our coefficient is 1. If n is even and $k = n/2$ our coefficient is -1.

When n is odd, the amounts of kasha converge to the same number for every person. If n is even the amounts of kasha converge to two numbers a and b, alternating between people. After each stealing, the amounts of kasha one bad-mannered person has fluctuate between a and b.

What can we conclude? It doesn't pay to be greedy and Schur's Lemma is in the center of representation theory.